Sound and Electromagnetic Waves and Optics

University Physics II-Part 2: Notes and exercises

Daniel Gebreselasie

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DANIEL GEBRESELASIE

SOUND AND ELECTROMAGNETIC WAVES AND OPTICS

UNIVERSITY PHYSICS II-PART 2: NOTES AND EXERCISES

Sound and Electromagnetic Waves and Optics: Part 2:
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12 WAVE MOTION

Your goal for this chapter is to understand the nature of waves, wave equation, harmonic waves and speed of a wave in a string.

A wave is a phenomenon in which a certain physical quantity varies as a function position and time resulting in the transfer of energy from one point to another through a medium. For example, for ocean waves, the physical quantity that varies as a function of position and time is the up and down vertical displacement of the water molecules resulting in the transmission of energy from the source say to the beaches. If a snap shot of the ocean is taken, a variation of the vertical displacement (y) of the water molecules as a function of position can be observed. The distance between two consecutive peaks of this variation is called the wavelength (λ) of the wave.

If we look at a particular point in the ocean, a molecule moving up and down as a function of time can be observed. The time taken for a certain molecule to make one complete up and down oscillation is called the period (T) of the wave. A peak of a wave travels a distance of one wavelength in one period. Therefore the speed (v) of a wave can be obtained as a ratio between the wavelength and period. That is $v = \lambda/T$. The number of cycles executed per second is called the frequency (f) of the wave. Since frequency is the inverse of the period ($T = 1/f$), the speed of the wave may also be given as

$$v = \lambda f$$

Example: Consecutive peaks of an ocean wave are separated by a distance of 3m. If the peaks of the wave are approaching the beach with a speed of 5m/s, calculate the time taken for the water molecules to complete one up and down oscillation.

Solution:

$$\lambda = 3 \text{ m};\ v = 5 \text{ m/s};\ T = ?$$

$$v = \frac{\lambda}{T} = \frac{3}{5} \text{ m/s} = 0.6 \text{ m/s}$$

The Wave Equation

The wave equation is an equation that results from application of Newton's second law (F=ma) to the particles through which the wave is propagating. If y is the physical quantity that varies as a function of position and time and the wave is travelling along the x-axis, the wave equation can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

As will be shown later, v is the speed of the wave. The solution of the wave equation is any function whose argument is $x \pm vt$; that is, $y = f(x \pm vt)$.

Example: For any function f, by direct substitution, shown that $f(x - vt)$ is a solution of the wave equation.

Solution:

Let $x - vt = u$, then $f(x - vt) = f(u)$, $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial u}{\partial t} = -v$. With

$y = f(u)$, $\frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial x}\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f(u)}{\partial t}\right)$. But $\frac{\partial f(u)}{\partial x} = \frac{df(u)}{dx}\frac{\partial u}{\partial x} = \frac{df(u)}{du}$ and

$\frac{\partial f(u)}{\partial t} = \frac{df(u)}{du}\frac{\partial u}{\partial t} = -v\frac{df(u)}{du}$. Therefore $\frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{df(u)}{du}\right) = \frac{d}{du}\left(\frac{\partial f(u)}{\partial x}\right) = \frac{d^2 f(u)}{du^2}$; and $\frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f(u)}{\partial t}\right) = \frac{1}{v^2}\frac{d}{dt}\left(-v\frac{\partial f(u)}{\partial u}\right) = \frac{-1}{v}\frac{d}{du}\left(\frac{\partial f(u)}{\partial t}\right) = \frac{d^2 f(u)}{du^2}$ proving the wave equation is satisfied.

f(x –vt) and f(x + vt) as Travelling Wave

$f(x \pm vt)$ represents the value of the wave, y, at location x after a time inverted t. The value of the function corresponding to a constant argument will be constant. But for the argument to be a constant, $x \pm vt$ must be constant. For $x \pm vt$ to be constant x must be changing with time because time is always increasing. This means the position of the value of the function corresponding to a certain constant argument must be changing with time. In other words it must be travelling. For example the peak of an ocean wave is a constant corresponding to a constant argument of the function. But for the argument $x \pm vt$ to be constant, x must be changing with time because time is increasing. This implies the peak of the wave must be travelling with time which we know is the case.

The speed of a wave is defined to be the speed of a certain constant value of the function which can be taken to be the peak of the wave. But this necessitates that the argument $x \pm vt$ must be constant;

$x \pm vt = constant$. The speed of the wave is equal to $\frac{dx}{dt}$. Taking the derivative of both sides

$$\frac{dx}{dt} \pm v = 0 \quad \Rightarrow \quad \frac{dx}{dt} = \pm v$$

This shows that the variable v in the wave equation actually represents the speed of the wave. This also shows that the argument $x + t$ represents a wave moving to the left (negative velocity) and the argument $x - vt$ represents a wave moving to the right (positive velocity). That is, $f(x - vt)$ is a wave moving to the right and $f(x + vt)$ is a wave moving to the left.

Example: A stone is dropped in water initiating a disturbance that satisfies the wave equation $\frac{\partial^2 y}{\partial x^2} = 4\frac{\partial^2 y}{\partial t^2}$ where y is the vertical displacement (disturbance) of the water molecules. How far would the disturbance travel in 5 seconds?

Solution:

$\Delta t = 5\ s;\ \Delta x = ?$

Comparing the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ with this particular wave equation we see that $\frac{1}{v^2} = 4$ and$v = 0.5$ m/s. Since the disturbance is a constant value of the wave (travels as it is), its argument must be constant; That is $x - vt = constant$. Taking the change of both sides $\Delta x - v\Delta t = 0$ (change of a constant is zero*)* and $\Delta x = 0.5 \times 5\ \text{m} = 2.5\ \text{m}$ The disturbance will be 2.5 m away after 5 s.

Harmonic (Sinusoidal) Wave

A typical wave in nature is sinusoidal wave which is a wave where the function f has a cosine or a sine form. So far we have considered a function whose argument is distance ($x \pm vt$) But the cosine and sine functions take radians (unit less) as arguments. So the argument $x \pm vt$ needs to be modified to become unit less. Let's consider the argument $kx - \omega t$ where k has units of 1/m and ω has units of 1/*s*.

Further since the periodicity of a cosine or sine function is 2π, we require that $kx = 2\pi$ when $x = \lambda$ (the wavelength) and $\omega t = 2\pi$ when $t = T$ (the period). Therefore it follows that

$$k = \frac{2\pi}{\lambda}$$

$$\omega = \frac{2\pi}{T} = 2\pi f$$

k is called the wavenumber of the wave and ω is the angular frequency of the wave.

Therefore the solution of the wave equation in terms of a cosine function can be written as

$$y = A\cos(kx - \omega t)$$

A represents the maximum value of the wave and is called the amplitude of the wave.

A given value of the function (say the peak of the wave) occurs when the argument is a constant: $kx \pm \omega t = constant$. Taking the derivative to obtain $\frac{dx}{dt}$ (speed) $k\frac{dx}{dt} \pm \omega = 0$. Therefore

$$\frac{dx}{dt} = v = \mp\frac{\omega}{k}$$

That is the speed of the wave can also be expressed as the ratio between the angular frequency and the wave number. The negative sign (corresponding to $y = A\cos(kx + \omega t)$) represents a wave travelling to the left. And the positive sign (corresponding to $y = A\cos(kx - \omega t)$) represents a wave travelling to the right. It can be shown by direct substitution that $y = A\cos(Kx \pm \omega t)$ is a solution to the wave equation wit $v^2 = \frac{\omega^2}{k^2}$. The displacement of the peak of the wave (or other value) can be obtained as speed times time; that is $\int_{x_i}^{x_f} dx = \pm\frac{\omega}{k}\int_0^t dt$ and

$$\Delta x = \pm\frac{\omega}{k}t = \pm vt$$

Example: A certain harmonic wave varies as a function of position and time according to the equation$y = 10\cos(10x - 20t)$.

a) Calculate the distance between two consecutive peaks of the wave.

Solution:

$k = 10$ m; ω=20 rad/s; λ=?

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{10}\ \text{m} = 0.2\pi\ \text{m}$$

b) Calculate the time taken for one complete up and down oscillation of the molecules.

Solution:

$$T = ?$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10}\text{ s} = 0.1\pi\text{ s}$$

c) Is the wave moving to the right or to the left?

Solution: Since the argument is $10x - 20t$, it is going to the right.

d) Calculate the speed of the wave.

Solution:
$v = ?$

$$v = \frac{\omega}{k} = \frac{20}{10}\text{ m/s} = 2\text{ m/s}$$

e) Calculate the distance travelled by a peak of the wave in 10 seconds.

Solution:

$$\Delta x = vt = 2\times 10\text{ m} = 20\text{ m}$$

Example: A disturbance at a certain point resulted in the following two waves: $y_1 = 4\cos(5x - 20t)$ and $y_2 = 4\cos(5x + 20t)$. Calculate the distance between the disturbances carried by the two waves after 4 seconds.

Solution:

$y_1 = 4\cos(5x - 20t)$ represents a wave travelling to the right. After 4 seconds the disturbance travels a distance of $\Delta x_1 = \frac{\omega}{k}t = \frac{20}{4}\times 4\text{ m} = 20$ to the right. $y_2 = 4\cos(5x + 20t)$ represents a wave travelling to the left. Therefore the disturbance would have travelled a distance of $\Delta x_2 = \frac{\omega}{k}t = \frac{20}{4}\times 4\text{ m} = 20\text{ m}$ to the left. Therefore the distance between both disturbances after 4 seconds is $\Delta x_1 + \Delta x_2 = (20 + 20)\text{ m} = 40\text{ m}$.

Practice Quiz 12.1

Choose the best answer

1. Consecutive peaks of an ocean wave are separated by a distance of 3 m. If the peaks of the wave are approaching the beach with a speed of 5 m/s, how many up and down oscillations do the water molecules execute per second?
   A. 1.5 Hz
   B. 1.167 Hz
   C. 2.333 Hz
   D. 1.667 Hz
   E. 2 Hz

2. A certain harmonic motion varies with time and position according to the equation $y = 0.4$ m $\cos(50t - 95x)$. Calculate the speed of the wave.
   A. 0.684 m/s
   B. 0.474 m/s
   C. 0.737 m/s
   D. 0.632 m/s
   E. 0.526 m/s

3. A certain wave $y(x, t)$ satisfies the wave equation $\partial^2 y/\partial x^2 = 0.023\, \partial^2 y/\partial t^2$ Calculate the speed of the wave.
   A. 5.934 m/s
   B. 6.594 m/s
   C. 4.616 m/s
   D. 5.275 m/s
   E. 2.638 m/s

4. Which of the following is a possible solution of the wave equation $\partial^2 y/\partial x^2 = 0.063\, \partial^2 y/\partial t^2$
   A. $y(x, t) = (x - 3.984t)^2 + x^4$
   B. $y(x, t) = (x - 3.984t)^2$
   C. $y(x, t) = (x - 15.873t)^2$
   D. $y(x, t) = (x2 - 3.984t)^2$
   E. $y(x, t) = (x - 15.873t2)^2$

5. A traveling wave initiated at the origin of an x-axis varies on position and time according the equation $y\ (x,\ t) = 4/(x - 3t)$ Where would the value of the wave at $t = 8$ s be the same as the value of the wave at the origin at $t = 0.5$ s?
   A. 22.2 m
   B. 22.35 m
   C. 22.5 m
   D. 23.4 m
   E. 22.05 m

6. Which of the following is a possible solution of the wave equation $\partial^2 y/\partial x^2 = 0.048\ \partial^2 y/\partial t^2$ that is moving to the right?
   A. $y\ (x,\ t) = (x - 20.833t)^2$
   B. $y\ (x,\ t) = (x - 4.564t)^2$
   C. $y\ (x,\ t) = (x + 20.833t)^2$
   D. $y\ (x,\ t) = (x - 0.048t)^2$
   E. $y\ (x,\ t) = (x + 4.564t)^2$

7. Calculate the distance travelled by a traveling wave in *15.3* seconds if the wave satisfies the wave equation $\partial^2 y/\partial x^2 = 0.035\ \partial^2 y/\partial t^2$
   A. *81.782* m
   B. *106.317* m
   C. *73.604* m
   D. *98.138* m
   E. *130.851* m

8. The maximum value of a certain harmonic wave is *0.6* m. Consecutive peaks of the wave are separated by a distance of *4.8* m. If the wave satisfies the wave equation $\partial^2 y/\partial x^2 = 0.023\ \partial^2 y/\partial t^2$ obtain an expression for the wave as a function of position and time.
   A. $y\ (x,\ t) = 0.6$ m $\cos\ (1.44x - 8.631t)$
   B. $y\ (x,\ t) = 0.6$ m $\cos\ (2.094x - 6.905t)$
   C. $y\ (x,\ t) = 0.6$ m $\cos\ (1.309x - 8.631t)$
   D. $y\ (x,\ t) = 0.6$ m $\cos\ (1.309x - 6.905t)$
   E. $y\ (x,\ t) = 0.6$ m $\cos\ (1.44x - 11.221t)$

Mechanical Energy of a Harmonic Wave

As shown in a previous chapter, the displacement of a particle undergoing a harmonic oscillation is given by $y = A\cos(\omega t - \varphi)$ where ϕ is the phase angle and its mechanical energy is given by $E = \frac{1}{2}m\omega^2 A^2$, where *m* is the mass of the particle. In a harmonic wave, even though the disturbance is travelling, the particles of the medium carrying the wave are not travelling. Actually they are just oscillating back and forth like a harmonic oscillator. For example in water waves, even though the peak of the wave is moving away from the source, the wave molecules are just moving up and down like a harmonic oscillator.

Consider a particle at a certain location ($x = constant$) in a medium where a harmonic wave is travelling. The particle is oscillating back and forth according to the equation

$$y = A\cos(kx \pm \omega t) \text{ or } y = A\cos(\omega t \pm kx)$$

with kx as its phase angle. Therefore all the particles of the medium will have the same mechanical energy even though they might have different phase angles (because of different locations or x). The mechanical energy of a particle of mass d*m* at any location through which a harmonic wave is travelling is given by

$$dE = \frac{1}{2}dm\omega^2 A^2$$

And of course the total energy of all the particles undergoing harmonic oscillation can be obtained by integrating this equation.

$$E_{total} = \frac{1}{2} M\omega^2 A^2$$

Where M is the total mass of all of the particles. But a more interesting quantity in dealing with waves is the rate of transmission of energy-which is the amount of energy that crosses perpendicular cross-sectional area per a unit time. This is referred as transmission power (P). The average transmission power ($\bar{P}$) can be obtained as the ratio between the amount of energy (E_λ) that crosses a perpendicular cross-sectional area in one cycle (wavelength) to the time taken for one cycle (period).

$$\bar{P} = \frac{E_\lambda}{T}$$

E_λ may be obtained by integrating dE over an interval equal to one wavelength $E_\lambda = \int_x^{x+\lambda} dE = \int_x^{x+\lambda} \frac{1}{2} dm\omega^2 A^2$ For simplicity, let's consider a harmonic wave travelling in a uniform string. If μ is the mass per unit length of the string, then $dm = \mu dx$ and $E_\lambda = \int_x^{x+\lambda} \frac{1}{2}\mu\omega^2 A^2 dx = \frac{1}{2}\mu\omega^2 A^2 \int_x^{x+\lambda} dx$. But $\int_x^{x+\lambda} dx = \lambda$ and $\bar{P} = \frac{E_\lambda}{T} = \frac{\frac{1}{2}\mu\omega^2 A^2 \lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 \left(\frac{\lambda}{T}\right)$. And since $\frac{\lambda}{T} = v$ it follows that

$$\bar{P} = \frac{1}{2}\mu\omega^2 A^2 v$$

The rate of transmission of energy is proportional to the square of the angular frequency (ω), to the square of the amplitude (A) and to the speed of the wave (v).

Example: A harmonic wave of the form $y = 0.01\cos(10x - 500t)$ is travelling in a string whose mass per unit length is 0.004 kg/m. Calculate the amount of energy that crosses a certain point of the string per a unit time.

Solution:

$$\mu = 0.004 \text{ kg/m};\ A = 0.01 \text{ m};\ \omega = 500 \text{ rad/s};\ k = 10 \text{ 1/m};\ \bar{P} = ?$$

$$v = \frac{\omega}{k} = \frac{500}{10} \text{ m/s} = 50 \text{ m/s}$$

$$\bar{P} = \frac{1}{2}\mu\omega^2 A^2 v$$

$$=\frac{1}{2}(0.004)(500)^2(0.01)^2(50)\text{ Watt}$$

$$=2.5\text{ Watt}$$

$$\mathrm{A}(x,y)\text{ and }\mathrm{B}(x+\Delta x,y+\Delta y)$$

Speed of a Wave in a String

Consider a small part of a wave carrying string whose end points are located at the points $\mathrm{A}(x,y)$ and $\mathrm{B}(x+\Delta x,y+\Delta y)$. The speed of a wave in a string can be obtained by comparing the general form of the wave equation, with the wave equation obtained by applying Newton's 2nd law to a small mass element of the string. A small mass element of a string is subjected to tension (T) forces from both of its ends. The net vertical force is equal to $T\sin\theta_B - T\sin\theta_A$ The tangents of θ_A and θ_B should be equal to the slopes of the string at the respective locations. Therefore

$$\tan\theta_B=\left.\frac{dy}{dx}\right|_{x+\Delta x}\quad \textit{and}\quad \tan\theta_A=\left.\frac{dy}{dx}\right|_{x}.\text{ Therefore}$$

$$F_{net}^{y}=T(\sin\theta_B-\sin\theta_A)=T\left(\sin\left[tan^{-1}\left(\left(\frac{dy}{dx}\right)_{x+\Delta x}\right)\right]-\sin\left[tan^{-1}\left(\frac{dy}{dx}\right)_x\right]\right)$$

For small angles and $tan^{-1}(x) \approx x$. Therefore, for small angles

$F^{y}_{net} \approx T\left(\sin\left[\left(\frac{dy}{dx}\right)_{x+\Delta x}\right] - \sin\left[\left(\frac{dy}{dx}\right)_{x}\right]\right)$. Again since $\sin x \approx x$ for small angles it follows that

$F^{y}_{net} \approx T\left(\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_{x}\right)$ Applying Newton's second law to the vertical motion

$F^{y}_{net} \approx T\left(\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_{x}\right) = dm\frac{\partial^2 y}{\partial t^2}$. If μ is the mass per unit length of the string, then $dm = \mu \Delta x$ and $\left(\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_{x}\right)/\Delta x = \frac{1}{\frac{T}{\mu}}\frac{\partial^2 y}{\partial t^2}$. But as Δx approaches zero, the left side becomes $\frac{\partial^2 y}{\partial x^2}$ resulting in the wave equation for a wave moving in a string.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

Comparing this with the general wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$, it follows that the speed of a wave in a string is given by

$$v = \sqrt{\frac{T}{\mu}}$$

Example: It is found that a harmonic wave of the form $y = 0.001\cos(5x - 400t)$ travels in a string when an object of mass 10kg hangs from it. Calculate the mass per unit length of the string.

Solution:
$m = 10\ kg; \quad A = 0.01\ m; \quad \omega = 400\ rad/s; \quad k = 5\ 1/m$

The tension in the string is equal to the weight of the hanging object

$$T = m|g| = 10 \times 9.8\ \text{N} = 98\ \text{N}$$

$$v = \frac{\omega}{k} = \frac{400}{5}\ \text{m/s} = 80\ \text{m/s}$$

$$\mu = \frac{T}{v^2} = \frac{98}{80^2}\ \text{kg/m} = 0.15\ \text{kg/m}$$

Reflection and Transmission of Wave

When the medium across which a wave is travelling changes some of the wave may be reflected and some of it may be transmitted. When a wave is reflected from a less dense medium, it is reflected without phase change.

When a wave is reflected from a denser medium, its phase angle changes by π radian or 180 degrees. In other words it is inverted

A transmitted wave does not undergo a phase change.

Types of Waves

There are two kinds of waves. They are called transverse and longitudinal waves. A transverse wave is a wave where the direction of movement of the particles carrying the wave is perpendicular to the direction of propagation of energy. An example is an ocean wave. While the wave travels parallel to the surface of the ocean, the water molecules vibrate up and down perpendicular to the direction of propagation of energy.

Longitudinal waves are waves where the direction of vibration of the particles carrying the wave is parallel to the direction of propagation of energy. An example is sound wave. As sound travels through a medium, the molecules of the medium vibrate back and forth in the direction of propagations of sound.

Practice Quiz 12.2

Choose the best answer

1. A certain harmonic wave varies with position and time according to the equation $y(x, t) = 0.2 \cos(40t - 2x)$ m. Give a formula for the velocity of the harmonic oscillation of a particle located at a distance of 0.81 m from the source (origin).
   A. $-8.8 \sin(40t - 1.62)$ m / s
   B. $-8 \sin(40t - 1.62)$ m / s
   C. $-4.8 \sin(40t - 1.944)$ m / s
   D. $8 \cos(40t - 1.944)$ m / s
   E. $8.8 \cos(40t - 1.134)$ m / s

2. A harmonic wave of the form $y(x, t) = 0.5 \cos(2x - 50t)$ m is traveling in water. If a picture is taken 0.34 s after the wave is initiated, then the picture will look like the graph of
   A. $0.5 \cos(2x - 20.4)$ m
   B. $0.5 \cos(2x - 18.7)$ m
   C. $0.5 \cos(2x - 11.9)$ m
   D. $0.5 \cos(2x - 17)$ m
   E. $0.5 \cos(2x - 23.8)$ m

3. A harmonic wave of the form $y\ (x,\ t) = 0.34\ cos\ (70x - 9t)$ m is traveling in water. Calculate the mechanical energy of one molecule of water. (Gram molecular weight of water is *18* g and Avogadro's number, number of molecules in one gram molecular weight, is *6.022e23*).
   A. *9.796e-26* J
   B. *12.595e-26* J
   C. *19.592e-26* J
   D. *13.994e-26* J
   E. *11.195e-26* J

4. A harmonic wave of the form $y\ (x,\ t) = 0.73\ cos\ (40t - 6x)$ m is traveling in a string whose mass per unit length is *0.0073* kg / m. Calculate the average rate of transfer of energy through a certain perpendicular area.
   A. *24.897* W
   B. *12.449* W
   C. *22.822* W
   D. *20.748* W
   E. *26.972* W

5. A string of length *1.46* m and mass *0.0034* kg is attached to a hanging object. If a wave in the string travels with a speed of *180* m / s, calculate the mass of the hanging object.
   A. *10.779* kg
   B. *10.009* kg
   C. *6.929* kg
   D. *9.239* kg
   E. *7.699* kg

6. A harmonic wave of the form $y\ (x,\ t) = 0.34\ cos\ (190t - 1.6x)$ m is traveling in a string whose mass per unit length is *0.0021* kg / m. Calculate the tension in the string.
   A. *17.768* N
   B. *23.691* N
   C. *20.729* N
   D. *32.575* N
   E. *29.613* N

7. Which of the following statements is a correct statement?
   A. There is no phase change when a wave is reflected from a denser medium.
   B. When a wave encounters a boundary between two mediums, it will always be either *100%* reflected or *100%* transmitted.
   C. There is a phase change of π radian when a wave is reflected from a less dense medium.
   D. There is a phase change of π radian when a wave is transmitted to a less dense medium.
   E. There is no phase change when a wave is transmitted to a denser medium.

8. Which of the following statement is correct?
   A. A transverse wave is a wave where the molecules of the medium carrying the wave are moving in a direction parallel to the direction of propagation of energy
   B. A longitudinal wave is a wave where the molecules of the medium carrying the wave are moving in a direction perpendicular to the direction of propagation of energy
   C. A wave moving in a string is a longitudinal wave.
   D. Sound is a longitudinal wave.
   E. Ocean wave is a longitudinal wave.

13 SOUND WAVES

Your goal for this chapter is to understand the properties of sound waves, harmonic sound waves, intensity of sound waves and Doppler's effect.

Sound waves are produced by vibrating objects. A vibrating object produces compressions and rarefactions in the molecules in its vicinity creating pressure that propagates through the medium. In sound waves, the physical quantity that varies as a function of position and time is the pressure exerted on the molecules of the medium. The molecules of the medium carrying the pressure move back and forth about their equilibrium positions in the direction of propagation of the pressure. The distance between two compression (maximum pressure location) at a given instant of time is called the wavelength (λ) of the wave.

The time taken for the molecules to make one complete oscillation is called the period (T). Therefore the speed of sound (v) may be given as $v = \frac{\lambda}{T}$.

Sound waves are longitudinal waves, because the molecules move back and forth in the direction of propagation of sound.

Relationship between Displacement of the Molecules and the Pressure

The source of pressure is the difference between the displacements of neighboring molecules (if all the molecules displaced by the same amount, it would be linear motion without additional pressure). Let the displacement of the molecules be denoted by $s(x)$.

Consider a sample of molecules in a cylindrical volume of cross-sectional area $A_{\perp}$ and length Δx. A change in the displacement between neighboring molecules will result in a compression change of volume of $A_{\perp}\Delta s(x)$.The Bulk modules (B) of the medium is defined as $B = -\frac{F/A}{\Delta V/V}$.

$\frac{F}{A}$ is the excess pressure ΔP corresponding to the compression. The initial volume before compression is $V = A_{\perp}\Delta x$ and the change in volume due to the difference in displacement between neighboring molecules is $\Delta V = A_{\perp}\Delta s(x)$. Therefore $B = -\frac{\Delta P}{\Delta V/V} \Rightarrow \Delta P = -B\frac{\Delta V}{V} = -B\frac{A_{\perp}\Delta s}{A_{\perp}\Delta x}$.

Taking the limit as Δx approaches zero gives the following expression for the excess pressure.

$$\Delta P = -B\frac{\partial s(x)}{\partial x}$$

It is a partial derivative because the process considered is a constant time process.

Speed of Sound in a Medium

Let's apply Newton's second law to a small mass element Δm. If the density of the medium is ρ, then $\Delta m = \rho A_{\perp} \Delta x$ and from Newton's second law, $F_{net} = \Delta m \cdot a$. But $a = \frac{\partial^2 s}{\partial t^2}$. Therefore $F_{net} = \Delta m \frac{\partial^2 s}{\partial t^2}$. And substituting for Δm gives the expression $F_{net} = \rho A_{\perp} \Delta x \frac{\partial^2 s}{\partial t^2}$. The force on this mass element is caused by the pressure of both sides: $F_{net} = -\{\Delta P(x + \Delta x) - \Delta P(x)\}A_{\perp}$. Where $\Delta P(x + \Delta x)$ and $\Delta P(x)$ are pressure on the left and right sides of the mass element respectively.

The negative sign is needed because the direction of the force due to $\Delta P(x + \Delta x)$ is to the left.

Replacing the net force in Newton's second law with the expression in terms of pressure and taking the limit as Δx approaches zero results in the following equation for the rate of change of the excess pressure with position:

$$-\frac{\partial}{\partial x}\Delta P = \rho \frac{\partial^2 s}{\partial t^2}$$

And substituting for ΔP, using the equation $\Delta P = -B\frac{\partial s}{\partial x}$, gives the equation

$$\frac{\partial^2 s}{\partial x^2} = \frac{\rho}{B}\frac{\partial^2 s}{\partial t^2}$$

But this equation is the wave equation for sound. Comparing it with the general wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$, gives the following expression for the speed of sound in a medium whose density and bulk modulus are ρ and B respectively.

$$v = \sqrt{\frac{B}{\rho}}$$

Example: Density of water is 1000 kg/m^3. Its bulk modulus is 0.12×101^{10}Pa. Calculate the speed of sound in water.

Solution:

$$\rho = 1{,}000 \text{ kg/m}^3; \quad B = 0.21\times10^{10}\,\text{Pa}\,; v = ?$$

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{0.21\times10^{10}}{1{,}000}} \text{ m/s} = 1.45\times10^3 \text{ m/s}$$

Harmonic Sound Wave

Since the displacement of the molecules (*s*) satisfies the wave equation, the harmonic wave solution may be written as

$$s(x,t) = S_{max}\cos(kx - \omega t)$$

Where $S_{\max}$ is the maximum displacement of the molecules (amplitude). Since $\Delta P = -B\frac{\partial s}{\partial x}$, substituting the harmonic wave solution gives the following expression for the excess pressure

$$\Delta P = BS_{max}k\sin(kx - \omega t)$$

Therefore the maximum value of the pressure (ΔP_{max}) is related to the maximum displacement of the molecules by $\Delta P_{\max} = BS_{\max}k$. Substituting for k and B from the equations $k = \frac{\omega}{v}$ and $v = \sqrt{\frac{B}{\rho}}$ respectively, gives the following relationship between the maximum excess pressure and the maximum displacement of the molecules

$$\Delta P_{max} = \rho\omega v S_{max}$$

The amplitude of the pressure is proportional to the density of the medium (ρ), frequency of the medium (ω), speed of sound and the amplitude of the displacement of the molecules.

Example: Sound is travelling in a certain medium of density 2000 kg/m^3. If the displacement of the molecules vary as function of position and time according to the equation $s(x,y) = 10^{-6}\cos(0.4x - 160t)$.

a) Calculate the maximum value of the pressure.

Solution:

ρ = 2,000 kg/m$^{3;}$ k = 0.4 1/m; ω = 1,600 1/s; S_{max} = 10^{-6} m

$$v = \frac{\omega}{k} = \frac{160}{0.4}\ \text{m/s}\ = 4000\ \text{m/s}$$

$$v = \frac{\omega}{k} = \frac{160}{0.4}\ \text{m/s}\ = 4000\ \text{m/s}$$

$$\Delta P_{\max} = (2{,}000)(160)(4{,}000)(10^{-6})\ \text{Pa} = 1280\ \text{Pa}$$

b) Give a formula for the pressure as a function of position and time.

Solution:

If

$$s(x,t) = S_{max}\cos(kx - \omega t)$$

Then

$$\Delta P = BS_{max}k\sin(kx - \omega t)$$

Therefore

$$\varDelta P(x,t) = 1280\ \text{Pa}\ \cos(0.4x - 160t)$$

The Dependence of speed of Sound in Air on Temperature

Since the kinetic energy of the molecules is proportional to temperature, the speed of sound in air is proportional to the square root of the temperature in degree Kelvin. That is, the ratio of speed of sound (v), to the square root of temperature (T) is a constant. Therefore if the speeds of sound at temperatures T_1 and T_2 are v_1 and v_2 respectively, it follows that $\frac{v_1}{\sqrt{T_1}} = \frac{v_2}{\sqrt{T_2}}$. The speed of sound in air at 0°C is 331m/s. Therefore with $v_2 = 331$ m/s and $T_2 = 0$ °C $= (0 + 273)$ °K $= 273$ °K, the speed of sound as a function of temperature may be given as

$$v = 331 \text{ m/s} \sqrt{\frac{T}{273 \text{ °K}}}$$

Example: Calculate the wavelength of sound produced by a 500 Hz tuning fork at a temperature of 20°C.

Solution:

$$f = 500 \text{ Hz};\ T = 20\ ^\circ\text{C} = \left(20 + 273\right)\ ^\circ\text{K} = 293\ ^\circ\text{K};\ \lambda = ?$$

$$v = 331 \text{ m/s}\ \sqrt{\frac{T}{273\ ^\circ\text{K}}} = 331\sqrt{\frac{293}{273}}\ \text{m/s} = 343 \text{ m/s}$$

$$\lambda = \frac{v}{f} = \frac{343}{500}\ \text{m} = 0.68 \text{ m}$$

Practice Quiz 13.1

Choose the best answer

1. Which of the following statements is incorrect?
   A. The square of the speed of sound is inversely proportional to the density of the medium through which sound is traveling.
   B. The square of the speed of sound is proportional to temperature.
   C. The speed of sound is proportional to the bulk modulus of the medium through which sound is traveling.
   D. For sound waves, the physical quantity that varies as a function of position and time is the pressure of the molecules.
   E. Sound is a longitudinal wave.

2. Sound is travels in a medium of bulk modulus *6.2e10* Pa with a speed of *7.1e3* m / s. Calculate the density of the medium.
   A. *737.949* kg / m³
   B. *1106.923* kg / m³
   C. *1229.915* kg / m³
   D. *1475.898* kg / m³
   E. *860.94* kg / m³

3. A sound wave traveling in a medium of bulk modulus *1.3e10* Pa satisfies the wave equation $\partial^2 y / \partial x^2 = 6.3\text{e-}7\ \partial^2 y / \partial t^2$ Calculate the density of the medium.
   A. *5733* kg / m³
   B. *9009* kg / m³
   C. *8190* kg / m³
   D. *9828* kg / m³
   E. *11466* kg / m³

4. A certain sound is traveling in a medium of bulk modulus *1.3e10* Pa with a speed of *7.1e3* m / s. The molecules take *6.1e-3* s to make one complete oscillation. The maximum displacement of the molecules of the medium is *5.9e-12* m. Calculate the maximum pressure of the molecules.
   A. *1.224e-2* Pa
   B. *0.89e-2* Pa
   C. *1.113-2* Pa e
   D. *0.779e-2* Pa
   E. *0.668e-2* Pa

5. A certain harmonic sound wave is traveling in a medium of density *3.1e3* kg / m^3. The displacement of the molecules of the medium from their equilibrium positions varies on position and time according to the equation $s(x, t) = 5.7e\text{-}9 \cos(6.1e3t - 2.5x)$ m Calculate the maximum pressure of the molecules.
   A. *184.1* Pa
   B. *210.4* Pa
   C. *368.2* Pa
   D. *263* Pa
   E. *289.3* Pa

6. At what temperature would the speed of sound in air be *300* m / s?
   A. *-63.364* °C
   B. *-48.741* °C
   C. *-58.49* °C
   D. *-29.245* °C
   E. *-43.867* °C

7. A sound produced in air by a *710* Hz tuning fork has a wavelength of *0.5* m. Calculate the temperature of the medium (air).
   A. *53.332* °C
   B. *45.127* °C
   C. *57.434* °C
   D. *41.024* °C
   E. *32.82* °C

8. At what temperature would a harmonic sound wave of the form $\Delta P\ (x,\ t) = 12.1\ sin\ (690t - 2x)$ Pa travel in air?
   A. *16.507* °C
   B. *21.224* °C
   C. *18.866* °C
   D. *14.149* °C
   E. *23.582* °C

Intensity of Sound Waves (Harmonic Wave)

Intensity is defined to be amount of energy that crosses a unit perpendicular area per a unit time. Average intensity may be obtained as the energy that crosses a unit perpendicular area per one cycle. That is, the intensity (I) may be given as

$$I = \frac{E_\lambda}{A_\perp T}$$

Where E_λ is amount of energy contained in one wavelength (one cycle), T is period or time taken for one cycle and $A_\perp$ the area of the cross-section through which the wave is travelling.

At a given location each molecule is oscillating like a harmonic oscillator: $s(x,t) = S_{max}\cos(kx - \omega t)$ Therefore its total energy is the mechanical energy of a harmonic oscillator. For a small mass element dm contained in a path element dx, the energy dE may be given as $dE = \frac{dm}{2}\omega^2 S^2_{\text{max}}$. If the density of the meduim is ρ then $dm = \rho A_{\perp} dx$ and $dE = \frac{1}{2}\omega^2 S^2_{max} \rho A_{\perp} dx$.

Then, the amount of energy that crosses a cross-sectional area in one cycle (E_{λ}) may be obtained by integrating this from x to $x+\lambda$: $E_\lambda = \int_x^{x+\lambda} \frac{1}{2}\omega^2 S^2_{max}\rho A_{\perp} dx = \frac{1}{2}\omega^2 S^2_{max}\rho A_{\perp}(x+\lambda-x)$.

Therefore the amount of energy that crosses the cross-section in one cycle is given as

$$E_{\lambda} = \frac{1}{2}\omega^2 S^2_{max}\rho A_{\perp}\lambda$$

And the intensity is given as $I = \frac{E_\lambda}{A_\perp T} = \frac{\frac{1}{2}\omega^2 S^2_{max}\rho A_\perp \lambda}{A_\perp T} = \frac{1}{2}\omega^2 S^2_{max}\rho A_\perp\left(\frac{\lambda}{T}\right)$. But $\frac{\lambda}{T} = v$ (speed of the wave). Thus the average intensity may be given as

$$I = \frac{1}{2}\rho\omega^2 S^2_{max} v$$

Using the equation $\Delta P_{\text{max}} = \rho\omega v S_{max}$, the intensity may also be expressed in terms of the amplitude of the excess pressure (ΔP_{max}) as

$$I = \frac{\Delta P^2_{max}}{2\rho v}$$

Spherical waves

A very common type of sound wave is spherical wave. When sound is initiated at a certain point, it travels in all directions giving rise to a spherical wave. With transmission power (rate of transfer of energy) P given as $P = \frac{E}{T}$, the intensity of a spherical wave is given as

$$I_{sp} = \frac{P}{A_{\perp}}$$

For a spherical wave $A_{\perp}$ is the surface area of a sphere of radius r, where r is the distance from the source. Thus, with $A_{\perp} = 4\pi r^2$, the intensity of a spherical wave may also be given as

$$I_{sp} = \frac{P}{4\pi r^2}$$

The intensity of a spherical wave is inversely proportional to the square of the distance from the source.

Example: For a harmonic sound wave travelling through water, the water molecules are oscillating back and forth with a maximum displacement of 2×10^{-7} m. If consecutive compressions (maximum pressure points) are separated by a distance of 2 m $\left(\rho_w = 1000 \text{ kg/m}^3,\ B_w = 1.2\times10^9 \text{ Pa}\right)$

a) Calculate the amount of energy that crosses a unit perpendicular cross-sectional area per a unit time.

Solution:

$$S_{\max} = 2\times10^{-7} \text{ m};\ \lambda = 2 \text{ m};\ I = ?$$

$$I = \frac{1}{2}\rho\omega^2 S_{max}^2 v$$

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.1E9}{10E2}} \text{ m/s} = 1.45\times10^3 \text{ m/s}$$

$$\omega = kv = \left(\frac{2\pi}{\lambda}\right)v = \left(\frac{2\pi}{2}\right)(1.45E3) \text{ rad/s} = 4.55E3 \text{ rad/s}$$

$$I = \frac{1}{2}\rho\omega^2 S_{max}^2 v = \frac{1}{2}(1000)\left(4.55\times10^3\right)^2\left(2\times10^{-7}\right)_{max}^2\left(1.45\times10^3\right) = 0.13 \text{ w/m}^2$$

b) Calculate the pressure in the compressions

Solution:

$$\Delta P_{\max} = ?$$

$$I = \frac{\Delta P_{max}^2}{2\rho v}$$

$$\Delta P_{max} = \sqrt{2\rho v I} = \sqrt{2(1{,}000)\left(1.45\times10^3\right)(.013)} \text{ Pa} = 1.94\times10^2 \text{ Pa}$$

c) Calculate the amount of energy that crosses a unit perpendicular cross-sectional area in one cycle.

Solution:

$$\frac{E_\lambda}{A_\perp} = ?$$

$$I = \frac{E_\lambda}{A_\perp T} = \left(\frac{E_\lambda}{A_\perp}\right)\frac{1}{T}$$

$$\frac{E_\lambda}{A_\perp} = IT$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4.55\times10^{3}}\ \mathrm{s} = 1.38\times10^{-3}\,s$$

$$\frac{E_\lambda}{A_\perp} = IT = (.013)\left(1.33\times10^{-3}\right)\ \mathrm{J/m^2} \ = 1.794\times10^{-5}\ \mathrm{J/m^2}$$

Example: A loudspeaker of power 100 watts is producing spherical waves. Calculate the intensity of the sound waves at a distance of 10m from the speaker.

Solution:

$$P = 100\ \mathrm{W};\ r = 10\ \mathrm{m};\ I = ?$$

$$I = \frac{P}{4\pi r^2} = \frac{100}{4\pi 10^2}\ \mathrm{W/m^2} = \frac{1}{4\pi}\ \mathrm{W/m^2}$$

Example: The intensity due to a certain loudspeaker that produces spherical waves is found to be 104 W/m² at a distance 4m from the speaker. Calculate the intensity at a distance of 8m.

Solution:

$$r_1 = 4 \text{ m}; I_1 = 104 \text{ W/m}^2;\ r_2 = 4 \text{ m}; I_2 = ?$$

Since it is the same speaker

$$P_1 = P_2$$
$$I_1 r_1^2 = I_2 r_2^2$$
$$I_2 = \frac{I_1 r_1^2}{r_2^2} = \frac{10^4 (4)^2}{8^2} \text{ W/m}^2 = 2500 \text{ W/m}^2$$

Relationships between Properties of Sound and Human Sensation

The loudness of sound is related with the intensity of the sound. That is, the greater the intensity, the greater the loudness of the sound. But they are not related linearly but logarithmically. Loudness (β) in decibels is related with the intensity of sound as

$$\beta = 10 \log\left(\frac{I}{I_0}\right)$$

Where I_0 is the minimum intensity of sound that can be detected by the human ear. Its value is 10^{-12} W/m^2. By the way, the maximum intensity that can be tolerated by the human ear is 1 W/m^2.

The pitch of sound is related with the frequency of sound. The greater the frequency the greater the pitch.

Example: Consider spherical waves produced by a 50 watt speaker.

a) Calculate the loudness in decibels at a distance of 10 m from the speaker.

Solution:

$$P = 50 \text{ W}; r = 10 \text{ m}; \beta = ?$$
$$\beta = 10 \log\left(\frac{I}{I_0}\right)$$
$$I = \frac{p}{4\pi r^2} = \frac{50}{4\pi 10^2} \text{ W/m}^2 = 4\times 10^{-2} \text{ W/m}^2$$
$$\beta = 10 \log\left(\frac{4\times 10^{-12}}{1\times 10^{-12}}\right) \text{dB} = 106 \text{ dB}$$

b) At what distance from the speaker would the loudness of the sound be 80 dB?

Solution:

$$\beta = 80 \text{ dB}; \quad r = ?$$

$$\beta = 10 \log\left(\frac{I}{I_0}\right)$$

$$80 = 10 \log\left(\frac{I}{I_0}\right)$$

$$\log\left(\frac{I}{I_0}\right) = 8$$

$$\frac{I}{I_0} = 10^8$$

$$I = \left(10^8\right)\left(10^{-12}\right) \text{ W/m}^2 = 10^{-4} \text{ W/m}^2$$

$$I = \frac{P}{4\pi r^2}$$

$$r = \sqrt{\frac{P}{4\pi I}} = \sqrt{\frac{50}{4\pi(10)^{-4}}} \text{ m} = 199 \text{ m}$$

Doppler's Effect

Doppler's effect refers to the change in frequency (pitch) of the sound heard by an observer due to the speed of the observer or the source with respect to the medium carrying the sound wave which is air in this case. (Speed with respect to air basically means speed with respect to the ground). It is a common experience that the pitch of a sound heard by an observer due to the horn of a car moving towards the observer increases (and decreases when the car moves away from the observer).

Let f be the frequency of the sound produced by the source, f' be the modified frequency heard by the observer and let v, v_o and v_s be the speeds of sound, observer and source respectively with respect to the medium carrying the sound wave (air). And further let v_o (v_s) be positive when the observer (source) is moving in the direction of the sound heard by the observer and negative when the observer (source) is moving opposite to the direction of the sound heard by the observer.

There are two sources for Doppler's effect: the motion of observer and the motion of source. Let's first deal with them separately and then we will combine both effects.

If the source is stationary and observer is moving, the speed with which sound is approaching (going away from) changes. Actually the new speed will become v-v_o. Remember according to our sign convention, v_o is negative if the observer is approaching the source and positive if going away from the source. The wavelength of the sound wave (distance between consecutive peaks) remains the same because it does not depend on the motion of the observer. Therefore it follows that $f = \frac{v}{\lambda}$ and $f' = \frac{v - v_0}{\lambda}$. Substituting for λ in the expression for f we get

$$f' = f\left(\frac{v - v_0}{v}\right)$$

If source is moving and observer is stationary, the speed with which sound waves arrive the observer doesn't change (the speed of sound depends on the medium and not on that of the source) but the distance between consecutive peaks (*wavelength*) changes because the center from which the spherical sound waves spread is moving with the source. Since the center of the spherical wave moves a distance of $v_s T = \frac{v_s}{f}$ in one period (T), the wavelength will change from λ to $\lambda - \frac{v_s}{f}$. (Remember according to our sign convention, will be positive when source approaches observer and negative when source goes away from observer). Therefore $f' = \frac{v}{\lambda - \frac{v_s}{\lambda}}$ and $f' = \frac{v}{\lambda - \frac{v_s}{\lambda}}$. Substituting for λ in the expression for f', gives the equation

$$f' = f\left(\frac{v}{v - v_s}\right)$$

If both observer and source are moving, both effects can be combined as follows. Let f'' be the modified frequency due to observer moving and source stationary. Then $f'' = f\left(\frac{v - v_o}{v}\right)$. Now suppose observer is kept stationary and source is moving. Then $f' = f''\left(\frac{v}{v - v_s}\right)$. Substituting for f'', the general expression combing both effects is obtained

$$f'' = f\left(\frac{v - v_0}{v - v_s}\right)$$

The above two formulas are special cases of this formula corresponding to $v_s = 0$ and $v_0 = 0$ respectively.

Example: A stationary car is blowing its horn with a frequency of 1000 Hz. Calculate the frequency of the sound heard by a cyclist when temperature is 20°C.

a) When the cyclist is travelling towards the car with a speed of 10 m/s.

Solution:

$f = 1000\text{ Hz};\ v_s = 0;\ T = 20\ °\text{C} = 293\ °\text{K};\ v_0 = -10\text{ m/s}$ (Negative because he is moving opposite to the direction of the sound received); $f' = ?$

$$v = 331\sqrt{\frac{T}{293\ °\text{K}}}\ \text{m/s} = \sqrt{\frac{293}{273}}\ \text{m/s} = 343\ \text{m/s}$$

$$f' = f\left(\frac{v - v_0}{v - v_s}\right) = 1000\left(\frac{343 - (-10)}{343 - 0}\right)\ \text{Hz} = 1029\ \text{Hz}$$

b) When the cyclist is travelling away from the car with a speed of 10 m/s

Solution:
v_o = 10 m/s (positive because he is moving in the direction of the sound received)

$$f' = f\left(\frac{v - v_o}{v - v_s}\right) = 1000\left(\frac{343 - 10}{343 - 0}\right)\text{ Hz} = 971\text{ m/s}$$

Example: Calculate the frequency of the sound heard by a stationary observer (at a temperature of 25°C) due to a 200 Hz sound produced by a car.

a) When the car is travelling towards the observer with a speed of 40 m/s.

Solution:

$f = 500$ Hz; $v_0 = 0$; $T = 25\ °\text{C} = 298\ °\text{K}$; $v_s = 4$ m/s (Positive because the car is travelling in the direction of the sound received by the observer); $f' = ?$

$$v = \sqrt{\frac{T}{273}} = 331\sqrt{\frac{298}{273}}\text{ m/s} = 346\text{ m/s}$$

$$f' = f\left(\frac{v - v_o}{v - v_s}\right) = 500\left(\frac{346 - 0}{346 - 40}\right)\text{ Hz} = 565\text{ Hz}$$

b) When the car is travelling away from the observer with a speed of 40 m/s

Solution:
$v_s = -40$ m/s (negative because the car is moving opposite to the direction of the sound received by the observer)

$$f' = f\left(\frac{v - v_o}{v - v_s}\right) = 500\left(\frac{346 - 0}{346 - (-40)}\right)\text{ Hz} = 448\text{ Hz}$$

Example: A cyclist and a car are travelling in the same direction. The car is blowing its horn at a frequency of 800Hz. The cyclist is travelling at a speed of 10 m/s. The car is travelling at a speed of 40 m/s/ (Assume temperature is 20˚C). Calculate the frequency of the sound heard by the cyclist?

a) When the car is behind the cyclist.

Solution:

f = 800 Hz

v_0 = 10 m/s (positive because he is moving in the direction of the sound received)

v_s = 40 m/s (positive because the car is moving in the direction of the sound received)

T = 20°C = 293°K; $f' = ?$

$$v = 331\sqrt{\frac{T}{273}} = 331\sqrt{\frac{293}{273}}\ \text{m/s} = 343\ \text{m/s}$$

$$f' = f\left(\frac{v - v_o}{v - v_s}\right) = 800\left(\frac{343 - 10}{343 - 40}\right)\ \text{Hz} = 879\ \text{Hz}$$

b) When the car overtakes him and is travelling in front of him.

Solution:

v_0 = −10 m/s (negative because he is moving opposite to the direction of sound receives)

v_s = −40m/s (negative because the car is moving opposite to the direction of sound received)

$f' = ?$

$$f' = f\left(\frac{v - v_o}{v - v_s}\right) = 800\left(\frac{343 - (-10)}{343 - (-40)}\right)\ \text{Hz} = 737\ \text{Hz}$$

Practice Quiz 13.2

Choose the best answer

1. Which of the following is an incorrect statement?
   A. The property of sound wave related with the pitch of sound is the frequency of the wave.
   B. The property of sound wave related with loudness of sound is the amplitude of the wave.
   C. The unit of measurement for loudness of sound is deci bell.
   D. Doppler's effect is the change in the pitch of sound heard by an observer due to the the relative speed between the observer and the source of sound.
   E. Intensity of sound is defined to be sound energy per a unit time.

2. A certain harmonic sound wave is traveling in a medium of density *4362* kg / m^3 and bulk modulus *7.3e10* Pa. The maximum displacement of the molecules is *3.7e-10* m. Consecutive points of maximum pressure are separated by a distance of *1.4* m. Calculate the amount of energy that crosses a unit perpendicular area per a unit time.
   A. *0.576e-3* W / m^2
   B. *0.412e-3* W / m^2
   C. *0.329e-3* W / m^2
   D. *0.535e-3* W / m^2
   E. *0.453e-3* W / m^2

3. A certain harmonic sound wave is traveling in a medium of density *4362* kg / m^3 and bulk modulus *5.1e10* Pa. The maximum pressure of the molecules is *3.7e-2* m. Calculate the amount of energy that crosses a unit perpendicular area per a unit time.
   A. *0.367e-10* W / m^2
   B. *0.413e-10* W / m^2
   C. *0.643e-10* W / m^2
   D. *0.597e-10* W / m^2
   E. *0.459e-10* W / m^2

4. A loud speaker is producing spherical sound waves. If the intensity of the sound at a distance of 12 m is 0.6 W / m ^{2}, calculate the intensity of the sound at a distance of 5 m.
   A. 3.11 W / m ^{2}
   B. 4.147 W / m ^{2}
   C. 4.493 W / m ^{2}
   D. 2.074 W / m ^{2}
   E. 3.456 W / m ^{2}

5. A loud speaker is producing spherical sound waves. Calculate the power of the loud speaker if the intensity of the sound at a distance of 3 m from the loud speaker is 0.1 W / m ^{2}.
   A. 12.441 W
   B. 11.31 W
   C. 15.834 W
   D. 7.917 W
   E. 9.048 W

6. Calculate the loudness level of sound of intensity 0.2 W / m^2.
   A. 113.01 dB
   B. 135.612 dB
   C. 124.311 dB
   D. 67.806 dB
   E. 146.913 dB

7. A man on a bicycle is going away from a stationary car that is producing sound of frequency 900 Hz with a speed of 5 m / s at a day when the temperature is 35 °C. Calculate the frequency of the sound heard by the man.
   A. 1242.081 Hz
   B. 887.201 Hz
   C. 549.449 Hz
   D. 798.481 Hz
   E. 1064.641 Hz

8. A car that is producing sound of frequency *1500* Hz is approaching a stationary man with a speed of *35* m / s at a day when the temperature is *25*°C. Calculate the frequency of the sound heard by the man.
   A. *1668.906* Hz
   B. *1835.797* Hz
   C. *2002.687* Hz
   D. *1168.234* Hz
   E. *1001.344* Hz

9. A car and a cyclist are traveling towards each other (in opposite directions) with speeds of *44* m / s and *12* m / s respectively. The car is producing sound of frequency *1400* Hz. If the temperature is *25*°C, calculate the frequency of the sound heard by the cyclist.
   A. *1493.779* Hz
   B. *2157.681* Hz
   C. *1659.754* Hz
   D. *995.853* Hz
   E. *1327.803* Hz

14 SUPERPOSITION (INTERFERENCE) OF WAVES AND STANDING WAVES

Your goal for this chapter is to learn about the properties of interfering waves, standing waves and a beat.

Two waves are said to interfere if they act at the same location at the same time. When two or more waves act at the same position (same particle) at the same time, the net effect is obtained by adding the waves algebraically. If the waves $y_1, y_2, y_3, \ldots$ are acting at a certain position at the same time, then the net wave $\left(y_{net}\right)$ is obtained as

$$y_{net} = y_1 + y_2 + y_3 + \ldots$$

Superposition of Harmonic Waves with Different Phase Angles (but same frequency and wavelength)

At the point of interference (x = constant), the two waves become harmonic oscillations that depend on time only which can be represented as, $y_1 = A_1 \cos\left(\omega t - \beta_1\right)$ and $y_2 = A_2 \cos\left(\omega t - \beta_2\right)$. There are 3 factors that contribute to the phase angles (β_1 and β_2) of the harmonic oscillation at the point of interference. The first is the initial phase angle (φ) which is the argument of the wave at $t = 0$ and $x = 0$. The second is the difference between the distances travelled by the two waves by the time they reach the interference point. The third is the difference between the times the waves were initiated. Let the two waves be represented as $y_1 = A_1 \cos\left(\omega t_1 - kx_1 - \phi_1\right)$ and $y_2 = A_2 \cos\left(\omega t_2 - kx_2 - \phi_2\right)$.

Let the difference between the distance travelled by the two waves (commonly refereed as path difference) be denoted by Δx; that is $\Delta x = x_2 - x_1$ or $x_2 = \Delta x + x_1$. Let the difference between the times the two waves were initiated (commonly referred as time lag) be represented by Δt; that is $\Delta t = t_2 - t_1 \quad or \quad t_2 = \Delta t + t_1$. Now the two waves can be written as $y_1 = A_1 \cos\left(\omega t - kx_1 - \phi_1\right)$ and $y_2 = A_2 \cos\left(\omega\left(\Delta t + t_1\right) - k\left(x_1 + \Delta x\right) - \phi_2\right)$. Dropping the subscript 1, because it is not needed any more, these may be rewritten as $y_1 = A_1 \cos\left(\omega t - kx - \phi_1\right)$ and $y_2 = A_2 \cos\left(\omega t - kx - \left(\kappa\Delta x - \omega\Delta t + \phi_2\right)\right)$.

Therefore the phase angles of the harmonic oscillations at the point of interference are $\beta_1 = kx + \phi_1$ and $\beta_2 = kx + (\mathrm{k}\Delta x - \omega\Delta t + \phi_2)$. And the phase difference between the two harmonic oscillations at the point of interference is $\beta_2 - \beta_1 = kx + (\kappa\Delta x - \omega\Delta t + \phi_2) - (kx + \phi_1)$ which simplifies to

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + (\phi_2 - \phi_1)$$

Where Δx is the path difference, Δt is the time lag and $\phi_2 - \phi_1$ is the difference between the initial phase angles (arguments at t = 0 and x = 0). The net oscillation is obtained by adding the two oscillators algebraically: $y_{net} = A_1 \cos(\omega t - \beta_1) + A_2 \cos(\omega t - \beta_2)$. Expanding the cosines gives $y_{net} = A_1 (\cos\beta_1 \cos(\omega t) + \sin\beta_1 \sin(\omega t)) + A_2 (\cos\beta_2 \cos(\omega t) + \sin\beta_2 \sin(\omega t))$ which reduces to $y_{net} = (A_1 \cos\beta_1 + A_2 \cos\beta_2)\cos\omega t + (A_1 \sin\beta_1 + A_2 \sin\beta_2)\sin\omega t$. These two terms can be combined into one term by introducing two values A and δ defined according to the equations $A\cos\delta = A_1 \cos\beta_1 + A_2 \cos\beta_2$ and $A\sin\delta = A_1 \sin\beta_1 + \sin\beta_2$. With these substitutions, the expression for the net oscillation becomes $y_{net} = A\cos\delta\cos\omega t + A\sin\delta\sin\omega t$ which reduces to

$$y_{net} = A\cos(\omega t - \delta)$$

Therefore A represents the amplitude of the net oscillation and δ represents the phase angle of the net oscillation. An expression for A in terms of A_1, A_2, β_1 and β_2 can be obtained squaring the two defining equations for A and δ, and then adding.

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 cos(\beta_2 - \beta_1)}$$

Similarly, an expression for δ can be obtained by dividing one of the defining equations by the other.

$$\delta = tan^{-1}\left(\frac{A_1 sin\beta_1 + A_2 sin\beta_2}{A_1 cos\beta_1 + A_2 cos\beta_2}\right)$$

If the two oscillations have the same amplitude ($A_1 = A_2$), the expression for the amplitude of the net wave simplifies to

$$A = 2\left|A_1 cos\left(\frac{\beta_2 - \beta_1}{2}\right)\right|$$

Constructive Interference

Constructive interference is superposition of two waves resulting in the maximum possible amplitude. The maximum value of $A=\sqrt{A_1^2+A_2^2+2A_1A_2cos\left(\beta_2-\beta_1\right)}$ occurs when $cos(\beta_2-\beta_1)=1$ because the maximum value of cosine is one. Therefore for constructive interference. $A=\sqrt{A_1^2+A_2^2+2A_1A_2}=\sqrt{(A_1+A_2)^2}=|A_1+A_2|$. If the amplitudes are restricted to be positive (A negative amplitude can usually be converted to a positive by adding π to the phase angle), the amplitude of two waves interfering constructively is the sum of the amplitudes of the waves.

$$A_{\max}=A_1+A_2$$

The condition for constructive interference is that $\cos\left(\beta_2-\beta_1\right)=1$, which implies the phase difference should be an integral multiple of 2π; That is,

$$\beta_2-\beta_1=2n\pi$$

Where n is an integer. The phase difference $\beta_2-\beta_1$ is called the phase shift of wave 2 with respect to wave 1. And when the phase shift is an integral multiple of 2π, we say the two waves are in phase.

Destructive Interference

Destructive interference is superposition resulting in the minimum possible amplitude of the net wave. The minimum value of $A=\sqrt{A_1^2+A_2^2+2A_1A_2cos\left(\beta_2-\beta_1\right)}$ occurs when $\cos\left(\beta_2-\beta_1\right)=-1$ because the minimum value of cosine is -1. Therefore for destructive interference $A=\sqrt{A_1^2+A_2^2+2A_1A_2\left(-1\right)}=\sqrt{\left(A_1-A_2\right)^2}$ and the amplitude of the net wave is equal to the absolute value of the difference between the amplitudes.

$$A_{\min}=\left|A_1-A_2\right|$$

The condition for destructive interference is that $\cos\left(\beta_2-\beta_1\right)=-1$ which implies the phase difference should be an odd integral multiple of π.

$$\beta_2-\beta_1=\left(2n+1\right)\pi$$

Where n is an integer. When the phase difference is an odd integral multiple of π, we say the two waves are out of phase.

Example: Determine the net amplitude when the following pair of waves meet at the same point at the same time.

a) $y_1=2\cos\left(20x-30t-\frac{\pi}{2}\right)\quad y_2=3\cos\left(10x-30t-\pi\right)$

Solution

$$A_1=2;\ A_2=3;\ \beta_1=\frac{\pi}{2};\ \beta_2=\pi;\ A=?$$

$$A=\sqrt{A_1^2+A_2^2+2A_1A_2cos\left(\beta_2-\beta_1\right)}$$

$$=\sqrt{2^2+3^2+2\times2\times3cos\left(\pi-\frac{\pi}{2}\right)}=\sqrt{13}$$

b) $y_1=-4\cos\left(20x-50t-\frac{\pi}{3}\right)\quad y_2=2\cos\left(20x-50t\right)$

Solution: A negative amplitude can be converted to positive by adding π to the phase angle

$A_1=4;\ A_2=2;\ \beta_1=\pi+\frac{\pi}{3};\ \beta_2=0;\ A=?$

$$A=\sqrt{A_1^2+A_2^2+2A_1A_2cos\left(\beta_2-\beta_1\right)}$$

$$\sqrt{4^2+2^2+2\times4\times2cos\left(\frac{4}{3}\pi-0\right)}=2\sqrt{3}$$

Example: When two waves of the same amplitude meet at the same position and time, it is found that the amplitude of the net wave is half of their amplitude. What are the possible values for the phase shift (difference between their phase angles) between the waves?

Solution:

$$\left|cos\left(\frac{\beta_2-\beta_1}{2}\right)\right| = 0.5$$

$$cos\left(\frac{\beta_2-\beta_1}{2}\right) = 0.5 \text{ or } -0.5$$

$$\beta_2-\beta_1 = \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}$$

For the general solution $2n\pi$ where n is an integer should be added to each solution.

Example: When two waves of amplitude 4 and 5 meet at the same location and same time, what are the minimum and maximum amplitude of the net wave that can be obtained?

Solution:

$$A_1 = 4;\ A_2 = 5;\ A_{\max} = ?;\ A_{\min} = ?$$

$$A_{max} = A_1 + A_2 = 4+5 = 9$$

$$A_{min} = |A_1 - A_2| = |4-5| = 1$$

Example: Determine whether the following pair of waves will interfere constructively, destructively or neither.

a) $y_1 = 2\cos\left(kx-\omega t-\frac{\pi}{2}\right) \quad y_2 = 4\cos\left(kx-\omega t-\frac{3\pi}{2}\right)$

Solution:

$$\beta_1 = \frac{\pi}{2};\ \beta_2 = \frac{3\pi}{2};\ \beta_2-\beta_1 = ?$$

$$\beta_2-\beta_1 = \frac{3\pi}{2}-\frac{\pi}{2} = \pi$$

Destructive interference

b) $y_1 = -5\cos(20x - 30t);\quad y_2 = 3\cos(20x - 30t - \pi)$

Solution:

$$\beta_1 = \pi;\ \beta_2 = \pi;\ \beta_2 - \beta_1 = ?$$
$$\beta_2 - \beta_1 = \pi - \pi = 0$$

Constructive interference

c) $y_1 = 2\cos(30x - 50t)\quad y_2 = -3\cos\left(30x - 50t + \frac{\pi}{2}\right)$

Solution: The phase angle of the second should be increased by π to convert the negative amplitude to positive.

$$\beta_1 = 0;\ \beta_2 = -\frac{\pi}{2} + \pi = \frac{\pi}{2};\ \beta_2 - \beta_1 = ?$$
$$\beta_2 - \beta_1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Neither constructive nor destructive interference

Example: Two waves of the form $y_1 = 5\text{ m}\cos\left(3000t_1 - 10x_1\right)$ $y_2 = 3\text{ m}\cos\left(3000t_2 - 10x_2\right)$ were initiated at the same time but at different points. By the time the two waves meet at a certain point, wave 2 has travelled a distance of 0.2 m more than the distance travelled by wave 1.

a) Calculate the amplitude of the net oscillation at the point interference.

Solution:

$$A_1 = 5\text{ m};\ A_2 = 3\text{ m};\ \phi_1 = \phi_2 = 0;\ \Delta t = 0;\ \Delta x = 2\text{ m};\ \omega = 3000\text{ rad/s};\ k = 10\text{ 1/m};\ A = ?$$

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + \left(\phi_2 - \phi_1\right) = 10\times 0.2 - 3000\times 0 + \left(0 - 0\right)\text{ rad/s} = 2\text{ rad/s}$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\left(\beta_2 - \beta_1\right)}$$
$$= \sqrt{5^2 + 3^2 + 2(3)(5)\cos(2)}\text{ m} = 4.638\text{ m}$$

b) Calculate the phase angle of the net harmonic oscillation at the point of interference if the distance travelled by the first wave is 5m.

Solution:

$$x_1 = 5\text{ m};\ x_2 = x_1 + \Delta x = 5.2\text{ m};\ \delta = ?$$

$$\beta_1 = kx_1 + \phi_1 = 10\times 5 + 0\text{ rad} = 50\text{ rad}$$

$$\beta_2 = kx_2 + \phi_2 = 10\times 5.2 + 0\text{ rad} = 52\text{ rad}$$

$$\delta = \tan^{-1}\left(\frac{A_1\sin\beta_1 + A_2\sin\beta_2}{A_1\cos\beta_1 + A_2\cos\beta_2}\right)$$

$$\delta = \tan^{-1}\left(\frac{5\sin(50) + 3\sin(52)}{5\cos(50) + 3\cos(52)}\right) = 0.363$$

c) Give a formula for the net harmonic oscillation at the point of interference as a function of time.

Solution:

$$y_{net} = y_1 + y_2 = A\cos(\omega t - \delta) = ?$$

$$y_{net} = 4.268 \text{ m}\cos(3000t - 0.363)$$

Example: Two waves of the form $y_1 = 5\cos(3000t_1 - 10x_1)$ $y_2 = 3\cos(3000t_2 - 10x_2)m$ were initiated at the same point. Wave 1 was initiated 0.004s earlier than wave 2 calculate the amplitude of the net oscillation by the time they meet at a certain point (*actually any point*).

Solution:

$$\omega = 3000 \text{ rad/s};\ k = 10 \text{ 1/m};\ A_1 = 5 \text{ m};\ A_2 = 3 \text{ m};\ \Delta x = 0;\ \Delta t = 0.004 \text{ s};\ \phi_1 = \phi_2 = 0;\ A = ?$$

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + (\varphi_2 - \varphi_1) = 10\times 0 - 3000\times 0.004 + (0-0) = -102$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\beta_2 - \beta_1)}$$

$$= \sqrt{5^2 + 3^2 + 2(3)(5)\cos(-102)} \text{ m} = 6.7 \text{ m}$$

Example: Two waves of the form $y_1 = 5 \text{ m}\cos(3{,}000t_1 - 10x_1 + 0.5)$ $y_2 = 3 \text{ m}\cos(3{,}000t_2 - 10x_2 - 0.2)$ are initiated at different times and different points. Wave 2 travelled 0.4 m more than that of wave 1 by the time they meet at the point of interference. Wave 1 was initiated 0.002 s earlier than wave 2. Calculate the amplitude of the net harmonic oscillation at the point of interference.

Solution:

$$\omega = 3000 \text{ rad/s};\ k = 10 \text{ 1/m};\ A_1 = 5 \text{ m};\ A_2 = 3 \text{ m};\ \Delta x = 0.4 \text{ m};\ \Delta t = 0.002 \text{ s};\ \phi_1 = -0.5;\ \phi_2 = 0.2;\ A = ?$$

$$\beta_2 - \beta_1 = k\Delta x - \omega\Delta t + (\varphi_2 - \varphi_1)$$

$$= 10(0.4) - 3{,}000(0.002) + (0.2 - (-0.5)) = -1.3$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\beta_2 - \beta_1)}$$

$$= \sqrt{5^2 + 3^2 + 2(3)(5)\cos(-1.3) \text{ m}} = 6.48 \text{ m}$$

Practice Quiz 14.1

Choose the best answer

1. Which of the following is not a correct statement?
   A. If two waves of the same amplitude interfere destructively, then the net wave is zero.
   B. The net wave of two interfering waves is obtained by adding the two waves instantaneously.
   C. When two harmonic waves interfere destructively, the amplitude of the net wave is equal to the absolute value of the difference between their amplitudes.
   D. Two harmonic waves interfere destructively if the difference between the phase angles of the harmonic oscillators at the point of interference is an even integral multiple of π.
   E. Wave interference is the meeting of two waves at the same point.

2. The following diagram shows two interfering waves.

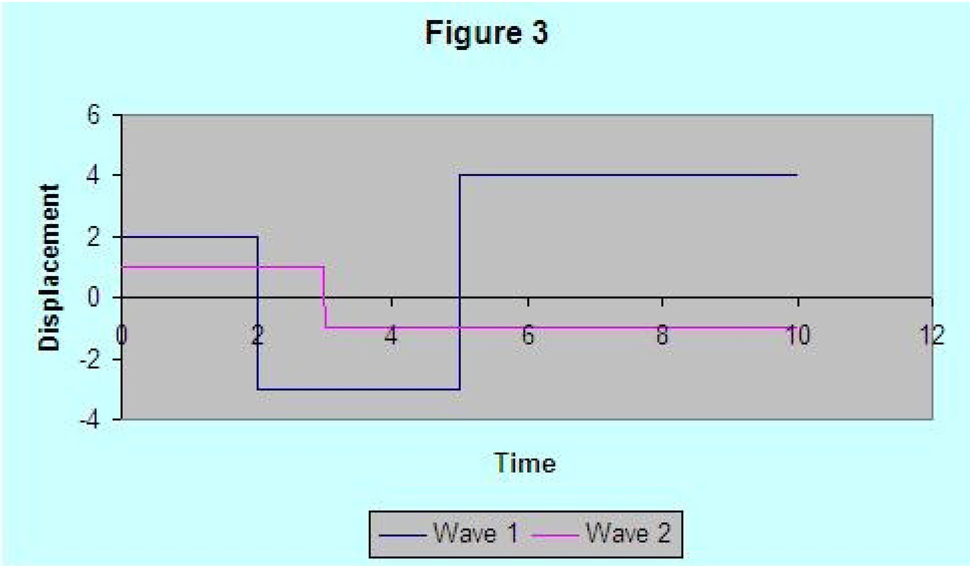

Figure 1

What is the value of the net wave in the first *2* seconds.

A. *-2* m
B. *2* m
C. *1* m
D. *3* m
E. *4* m

3. When two harmonic waves interfere at a certain point, the harmonic oscillations due to the waves at the point vary as a function of time according to the equations $y_1 = 8.5 \cos(20t - 0.6)$ m and $y_2 = 1.7 \cos(20t - 1.2)$ m. Calculate the amplitude of the net harmonic oscillation.

A. *9.949* m
B. *10.944* m
C. *12.934* m
D. *13.929* m
E. *11.939* m

4. When two harmonic waves interfere at a certain point, the harmonic oscillations due to the waves at the point vary as a function of time according to the equations $y_1 = 5.8 \cos(20t + 0.2)$ m and $y_2 = 9.3 \cos(20t + 0.6)$ m. Calculate the phase angle of the net harmonic oscillation.
A. -0.626 radians
B. -0.447 radians
C. -0.358 radians
D. -0.402 radians
E. -0.581 radians

5. When two harmonic waves interfere at a certain point, the harmonic oscillations due to the waves at the point vary as a function of time according to the equations $y_1 = 6.3 \cos(20t + 0.6)$ m and $y_2 = 3.4 \cos(20t + 0.1)$ m. Give a formula for oscillation of the net wave at the point as a function of time.
A. $y_{net} = 9.426 \cos(20t + 0.341)$ m
B. $y_{net} = 8.483 \cos(20t + 0.554)$ m
C. $y_{net} = 9.426 \cos(20t + 0.426)$ m
D. $y_{net} = 13.196 \cos(20t + 0.554)$ m
E. $y_{net} = 8.483 \cos(20t + 0.426)$ m

6. Calculate the amplitude of the net wave, when a wave of amplitude 1.5 m and a wave of amplitude 6.7 m interfere constructively.
A. 8.2 m
B. 0
C. 9.02 m
D. 9.84 m
E. 10.66 m

7. Which of the following pair of oscillations (due to 2 waves interfering at a certain point) interfere constructively?
A. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t - 2\pi)$
B. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t - 6\pi)$
C. $x = 10 \sin(30t - \pi)$ and $y = 20 \sin(30t + 3\pi)$
D. $x = 10 \sin(30t - \pi)$ and $y = 20 \sin(30t)$
E. $x = 10 \sin(30t - \pi/2)$ and $y = 20 \sin(30t + 5\pi/2)$

8. Which of the following pair of oscillations (due to 2 waves interfering at a certain point) interfere destructively?
   A. $x = 10 \sin (30t + \pi)$ and $y = 20 \sin (30t)$
   B. $x = 10 \sin (30t)$ and $y = 20 \sin (30t)$
   C. $x = 10 \sin (30t - \pi)$ and $y = 20 \sin (30t - 7\pi)$
   D. $x = 10 \sin (30t - \pi)$ and $y = 20 \sin (30t - 9\pi)$
   E. $x = 10 \sin (30t - \pi/2)$ and $y = 20 \sin (30t - 5\pi/2)$

9. When two harmonic waves of the same amplitude interfere at a certain point, the amplitude of the resulting net harmonic oscillation is found to be 0.1 times the amplitude of the waves. Which of the following is a possible phase difference between the two waves at the point of interference?
   A. 1.825 radians
   B. 3.042 radians
   C. 2.737 radians
   D. 2.129 radians
   E. 3.65 radians

10. Two waves vary with position and time according to the equations $y_1 = 0.3\cos(150t_1 - 2.6x_1 - 1.5)$ and $y_2 = 0.4\cos(150t_2 - 2.6x_2 - 2.4)$. The two waves were initiated at the same time but at different points. By the time the two waves meet at a certain point, the second wave has travelled *0.54* m more than the distance travelled by the first wave. Calculate the phase difference between the two waves at the point of interference.
    A. *2.765* radians
    B. *2.534* radians
    C. *2.995* radians
    D. *3.226* radians
    E. *2.304* radians

11. Two waves vary with position and time according to the equations
    $y_1 = 4.8\cos(100t_1 - 7.2x_1)$ m and
    $y_2 = 1.5\cos(100t2 - 7.2x2)$ m.

    The two waves were initiated at the same point. The first wave was initiated 0.45 s earlier than the second wave. Calculate the amplitude of the net oscillation at the point of interference.

    A. *5.732* m
    B. *6.305* m
    C. *8.025* m
    D. *7.451* m
    E. *4.586* m

Standing Waves

Standing waves are waves with equally spaced points of zero vibration. An example is the kind of wave that can be seen when a rubber band fixed at both sides is excited. The points of zero vibration are called nods. And the points of maximum vibration are called antinodes. A standing wave is usually formed when an incident wave and a reflected wave are superposed. Consider a wave of the form $y_i = A_1\cos(\omega t - kx - \phi)$ reflected from a boundary between two mediums (*or obstacles*) a distance of L from the source. When an incident wave and a reflected wave meet at a distance x from the source, the incident wave would have travelled a distance x and the reflected wave would have travelled a distance of $L + (L - x) = 2L - x$. As stated in the previous chapter, a reflected wave will have a phase change of π on reflection if the medium from which it is reflected is denser that it's medium and there will be no phase change if reflected from a less dense medium. Let's consider both separately.

Case 1: Wave Reflected from a Denser Medium

Let y_i and y_r be the incident and the reflected wave respectively. If $y_i = A_1 \cos(\omega t - Kx - \phi)$ then $y_r = A_2 \cos(\omega t - K(2L-x) - \phi - \pi)$.And the net wave y_{net} is given by $y_{net} = y_i + y_r = A_1 \cos(\omega t - Kx - \phi) + A_2 \cos(\omega t - K(2L-x) - \phi - \pi)$.

At a given location *x*, (*x* = *constant*) these two waves are harmonic

oscillators with phase angles $\beta_1 = Kx + \varphi$ *and* $\beta_2 = K(2L-x) + \varphi + \pi$. Therefore $y_{net} = A_1 \cos(Kx - \beta_1) + A_2 \cos(\omega t - \beta_2)$ which can be written as $y_{net} = A\cos(\omega t - \delta)$ with $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\beta_2 - \beta_1)}$ and $\delta = \tan^{-1}\left(\dfrac{A_1 \sin\beta_1 + A_2 \sin\beta_2}{A_1 \cos\beta_1 + A_2 \cos\beta_2}\right)$. Where $\beta_2 - \beta_1 = k(2L-x) + \phi + \pi - (kx + \phi) = 2k(L-x) + \pi$.

For simplicity, let's assume that the wave is reflected 100% (even

though some of it may be transmitted). Then $A_1 = A_2$. Therefore $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\beta_2 - \beta_1)} = \sqrt{2A_1^2 + 2A_1^2\cos(2k(L-x)+\pi)} = A_1\sqrt{2(1-\cos(2k(L-x)))}$. And using the trigonometric identity $\frac{1-\cos 2\theta}{2} = \sin^2\theta$, the following expression for the net amplitude as a function of distance is obtained.

$$A = 2A_1 \left|\sin(k(L-x))\right|$$

With this as an amplitude, the net wave is given by

$$y_{net} = 2A_1 \left|\sin(k(L-x))\right| \cos(\omega t - \delta)$$

This shows that the amplitude of the harmonic oscillators is a function of position, *x*. And since the amplitude varies like a sine, there are going to be points with zero amplitude or no vibration. These are the points called the nodes of the waves. And of course the points where sine is a maximum are the antinode points.

The nodes are the values of *x* for which the amplitude is zero. Let x_m be the m^{th} node with $x_0 = L$ being the reflection point. Then $\sin(k(L - x_m)) = 0$ which implies $k(L - x_m) = m\pi$ or

$$x_m = L - \frac{1}{2}m\lambda$$

The distance between consecutive node equal to $\left|x_{m+1}-x_m\right| = \left|\left(L-\frac{m+1}{2}\lambda\right)-\left(L-\frac{m}{2}\lambda\right)\right| = \frac{\lambda}{2}$. That is the size of one loop of the standing wave is half of the wavelength of the wave.

$$\text{length of one loop} = \left|x_{m+1}-x_m\right| = \frac{\lambda}{2}$$

At the point of reflection $x = L$. Therefore $A\big|_{x=L} = 2A_1\left|\sin K\left(L-L\right)\right| = 0$. That is, when a wave is reflected from a denser medium, the reflection point is a node.

Boundary conditions restrict the wavelengths of waves that can exist as standing wave in a medium. Two very common boundary conditions will be considered.

a) *The other end (x = 0) also required to be a node*: That is, the standing wave will have nodes at both ends. An example of this is a standing wave in a string where both ends are fixed. The amplitude at arbitrary value of x is given b $A = 2A_1 \left|\sin\left(k\left(L-x\right)\right)\right|$ (assuming 100% reflection). The condition $A|_{x=0} = 0$ implies that $\sin(kL) = \sin\left(\frac{2\pi}{\lambda}L\right) = 0$ or $\frac{2\pi}{\lambda_n}L = n\pi$ where n is a positive integer. Therefore if both ends are nodes only waves with wavelengths

$$\lambda_n = \frac{2L}{n}$$

Where n is a positive integer can form standing waves. In other words the only waves that can form standing waves in a standing wave of length L $\lambda_1 = \frac{2L}{1}$, $\lambda_2 = \frac{2L}{2} = L$, $\lambda_3 = \frac{2L}{3}$, $\lambda_4 = \frac{2L}{4} = \frac{L}{2}$, … and so on. Since the speed of a wave depends on the properties of the medium only, the frequencies that can exist as a standing wave are also restricted and the frequencies of the standing waves are given as $f_n = \frac{v}{\lambda_n}$ where v is the speed of the wave. But $\lambda_n = \frac{2L}{n}$. Therefore the only frequencies that can form standing waves are given by

$$f_n = n\left(\frac{v}{2L}\right)$$

Where n is a positive integer. The standing wave whose frequency is f_n called the n^{th} harmonic of the standing wave. The first harmonic is also called the fundamental harmonic. With $n = 1$, $f_1 = \frac{V}{2L}$. Therefore the frequency of the harmonic frequency may be expressed in terms of the fundamental frequency as

$$f_n = nf_1$$

This shows that the frequencies of all the harmonics of the standing wave are integral multiples of the fundamental frequency f_1. The part of the standing wave between two consecutive nodes is called a loop. The number of loops for the harmonic (N) can be obtained by dividing the length of the standing wave (L) by the size of one loop $\left(\frac{\lambda_n}{2}\right)$. Using the expression for λ_n in terms of L, it follows that $N = n$, that is, the n^{th} harmonic has n loops.

Example: A wave of the form $y = 5\cos(4\pi x - 20\pi t)$ is reflected from a boundary (*obstacle*) 2m away from the point where it is initiated. If the wave is reflected 100%.

a) Calculate the amplitude of the harmonic oscillation of a particle located at an antinode.

Solution:
At an antinode the amplitude is maximum: $A_1 = 5$

$$A = 2A_1 \left|\sin\left(k\left(L-x\right)\right)\right|$$

The maximum value of A occurs when $\sin[K(L-x)] = 1$ because maximum of a sine function is 1. Therefore

$$A_{max} = 2A_1 = 2(5) = 10 \text{ m}$$

b) Determine the size of one loop of the standing wave (*that is distance between consecutive nodes*).

Solution

$k = 4\pi$ 1/m; $|x_{m+1} - x_m| = ?$

$$k = \frac{2\pi}{\lambda} = 4\pi \text{ 1/m}$$

$$\lambda = 0.5 \text{ m}$$

$$\left|x_{m+1} - x_m\right| = \frac{\lambda}{2} = \frac{0.5}{2} = 0.25 \text{ m}$$

c) Determine location of the nodes.

Solution:

$L = 2$ m; $\lambda = 0.5$ m; $x_0, x_1, \ldots ?$

$$x_m = L - \frac{1}{2}m\lambda$$

$$x_o = 2 \text{ m};\ x_1 = 1.75 \text{ m};\ x_2 = 1.5 \text{ m};\ x_3 = 1.25 \text{ m};\ x_4 = 1 \text{ m};$$

$$x_5 = 0.75 \text{ m};\ x_6 = 0.5 \text{ m};\ x_7 = 0.25 \text{ m};\ x_8 = 0$$

d) Determine the number of loops.

Solution:
$\lambda = 0.5$ m; $N = ?$

Since the size of one loop is $\lambda/2$

$$N = \frac{L}{\left(\lambda/2\right)} = \frac{2L}{\lambda} = \frac{2(2)}{0.5} = 8$$

e) Calculate the amplitude of the harmonic oscillation of a particle located at x=0.4 m.

Solution:
$x = 0.4$ m; $A_{x=0.4} = ?$

$$A = 2A_1 \left|\sin\left(K\left(L - x\right)\right)\right|$$

$$A\big|_{x=0.4} = 2\left(5\text{ m}\right)\left|\sin\left(4\pi\left(2 - 0.4\right)\right)\right| = 9.5\text{ m}$$

Example: Consider a standing wave formed in a string of length $4m$ fixed at both of its ends. The mass of the string is 0.02 kg. There is a tension of 100 N in the string.

a) Determine the wavelengths of the first 3 harmonics.

Solution:

$L = 2\text{ m};\ \lambda_n = \frac{2L}{n} = ?$

$$\lambda_1 = \frac{2(4)}{1}\text{ m} = 8\text{ m}$$

$$\lambda_2 = \frac{2(4)}{2}\text{ m} = 4\text{ m}$$

$$\lambda_3 = \frac{2(4)}{3}\text{ m} = \frac{8}{3}\text{ m}$$

b) Determine the speed of the wave in the string.

Solution:

$T = 100\text{ N};\ l = 4\text{ m};\ m = 0.02\text{ kg};\ v = ?$

$$v = \sqrt{T/\mu}$$

$$\mu = \frac{m}{l} = \frac{0.02}{4}\text{ kg/m} = 0.005\text{ kg/m}$$

$$v = \sqrt{100/0.005}\text{ m/s} = 141\text{ m/s}$$

c) Calculate the wavelength and frequency of the 15th harmonic.

Solution:

$n = 15;\ f_{15} = ?;\ \lambda_{15} = ?$

$$f_1 = \frac{v}{\lambda_1} = \frac{141}{8}\text{ Hz} = 17.6\text{ Hz}$$

$$f_{15} = 15 f_1 = 15(17.6)\text{ Hz} = 264\text{ Hz}$$

$$\lambda_{15} = \frac{v}{f_{15}} = \frac{141}{264}\text{ m} = 0.534\text{ m}$$

b) *The x = 0 end required to be an Antinode*: Remember the other end (the reflection end) is a node. Therefore this is a standing wave with a node on one end and an antinode on the other end. An example of this is sound resonance in a tube closed on one end and open on the other end. It will have an antinode on the open end and a node on the closed end. Since $A = 2A_1\left|\sin\left(k\left(L-x\right)\right)\right|$, for an antinode, the value of the sine should be one because the maximum of sine is one. Therefore $\left(k\left(L-x\right)\right)\Big|_{x=0} = 1$ which implies that $k_n L = \left(\frac{2n-1}{2}\right)\pi$ where n is a positive integer. And with $k_n = \frac{2\pi}{\lambda_n}$, the following expression for the wavelength of the n^{th} harmonic can be obtained.

$$\lambda_n = \frac{4L}{2n-1}$$

This implies that the only waves that can exist in a standing wave of length L with a node on one end and an antinode on the other end are waves with wavelengths.

$$\lambda_1 = \frac{4L}{2(1)-1} = 4L, \quad \lambda_2 = \frac{4L}{2(2)-1} = \frac{4L}{3}, \quad \lambda_3 = \frac{4L}{2(3)-1} = \frac{4L}{5}, \quad \ldots \quad \text{and so on.}$$

If the speed of the wave is v (the speed depends only on the properties of the medium), then the frequency of the n^{th} harmonic is given by $f_n = \frac{v}{\lambda_n} = \frac{v}{\left(4L/(2n-1)\right)}$. Or

$$f_n = (2n-1)\frac{v}{4L}$$

Since, $f_1 = \frac{v}{4L}$, the frequency of the n^{th} harmonic may also be written in terms of the fundamental frequency μ_0 as

$$f_n = (2n-1)f_1$$

The number of loops in a standing wave may be obtained as the ratio of the length of the standing wave to the size of one loop (half of the wavelength).That is, $N = \frac{L}{\lambda_n/2} = \frac{L}{\frac{1}{2}\frac{4L}{2n-1}} = \frac{2n-1}{2}$.

The n^{th} harmonic has $\frac{2n-1}{2}$ loops when one end is a node and the other antinode.

Example: Consider sound resonance (*standing wave*) formed in a pipe open at one end closed at the other end. Its length is 0.5m. Assume the temperature is 20˚C.

a) Calculate the wave lengths of the first 3 harmonics.

Solution: The sound resonance will have a node at the closed end and antinode at the open end.

$L = 0.5$ m; $T = 20°\text{C} = 293°\text{K}$; $\lambda_n = ?$

$$\lambda_n = \frac{4L}{2n-1} = \frac{4(0.5)}{2n-1}\text{ m} = \frac{2}{2n-1}\text{ m}$$

$$\lambda_1 = \frac{2}{2(1)-1} = 2m;\quad \lambda_2 = \frac{2}{2(2)-1}\text{ m} = \frac{2}{3}\text{ m};\quad \lambda_3 = \frac{2}{2(3)-1}\text{ m} = \frac{2}{5}\text{ m}$$

$$v = 331\text{ m/s}\sqrt{\frac{T}{273°\text{K}}} = 331\sqrt{\frac{293}{273}}\text{ m/s} = 343\text{ m/s}$$

b) Calculate the frequencies of the first 3 harmonics.

Solution

$$v = 331 \text{ m/s} \sqrt{\frac{T}{273°\text{K}}} = 331\sqrt{\frac{293}{273}} \text{ m/s} = 343 \text{ m/s}$$

$$f_1 = \frac{v}{\lambda_1} = \frac{243}{2} \text{ Hz} = 172 \text{ Hz}$$

$$f_n = (2n-1) f_1$$

$$f_2 = (4-1)(172) \text{ Hz} = 516 \text{ Hz}; \; f_2 = (6-1)(172) \text{ Hz} = 860 \text{ Hz}$$

c) Calculate the wavelength and frequency of the 4th harmonic

Solution

$f_{11} = ?;\ \lambda_{11} = ?$

$$f_{11} = (2(11)-1) f_1 = (21)(172) \text{ Hz} = 3612 \text{ Hz}$$

$$\lambda_{11} = \frac{v}{f_{11}} = \frac{343}{3612} \text{ m} = 0.095 \text{ m}$$

Case 2: Wave Reflected from a less Dense Medium

In this case there is no phase shift on reflection. Therefore if the incident wave y_i is given by $y_i = A_1 \cos(\omega t - kx - \phi)$, Then the reflected wave is given by $y_r = A_2 \cos(\omega t - k(2L - x) - \phi)$.

As shown before the amplitude of the net wave is given by $A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\beta_2 - \beta_1)}$ where $\beta_1 = kx + \phi$ and $\beta_2 = k(2L - x) + \phi$. Therefore $\beta_2 - \beta_1 = 2K(L - x)$. And assuming the wave is reflected 100% $(A_1 = A_2)$, $A = \sqrt{A_1^2 + A_1^2 + 2A_1^2 \cos(2K(L-x))} = \sqrt{2}|A_1|\sqrt{1+\cos(2K(L-x))}$ $= \sqrt{2} A_1 \sqrt{2\cos^2(K(L-x))}$ $\left(\cos^2 x = \frac{1+\cos 2x}{2}\right)$. Thus the amplitude of the wave as a function of distance is given as

$$A = 2A_1 \cos(k(L-x))$$

At the reflection point $x = L$ and $A\big|_{X=L} = 2A_1 \cos(0) = 2A_1$ which is the maximum amplitude. Therefore the reflection points is an antinode.

Requiring the x = 0 point to be a node will result in a node in one end and an antinode in the other end which has been discussed already. So here only the case where the x = 0 end is required to be an antinode will be discussed. That is, a case where both ends are antinodes. An example is sound resonance formed in a pipe open at both ends. With a similar analysis, it can be shown that the equations for the case where both ends are antinodes are identical to the case where both ends are nodes: $\lambda_n = \frac{2L}{n}$, $f_n = n\left(\frac{v}{2L}\right)$, $f_1 = \frac{v}{2L}$, $f_n = nf_1$, $N = n$.

Example: Consider sound resonance (*standing waves*) in a 2 m pipe open at both ends. Assume temperature is 20°C. Calculate the wavelength and frequency of the 8th harmonic.

Solution:

$L = 2$ m; $T = 20°\text{C} = 293°\text{K}$; $\lambda_8 = ?; f_8 = ?$

$$f_1 = \frac{v}{2L}$$

$$v = \left(\sqrt{\frac{T}{273}}\right) 331 \text{ m/s} = 331\sqrt{\frac{293}{273}} \text{ m/s} = 343 \text{ m/s}$$

$$f_1 = \frac{343}{2(2)} \text{ Hz} = 85.7 \text{ Hz}$$

$$f_n = nf_1$$

$$f_8 = 8f_1 = 8(85.7) \text{ Hz} = 426.7 \text{ Hz}$$

$$\lambda_8 = \frac{v}{f_8} = \frac{343}{426.7} \text{ m} = 0.8 \text{ m}$$

A Beat

A beat is a wave with points of zero vibration which for a given location are separated by equal intervals of time; for example, for sound waves, at a given location, the event of no sound will be separated by equal intervals of time. The wave between two consecutive events of zero vibration is called a beat. The time taken for a beat is called the beat period and the number of beats per second is called the beat frequency. A beat is formed when two waves with close frequencies interfere. Consider two interfering waves $y_1 = A\cos(k_1 x - \omega_1 t)$ and $y_2 = A\cos(k_2 x - \omega_2 t)$ of frequencies ω_1 and ω_2 such that $\frac{|\omega_1 - \omega_2|}{\omega_1} < 1$:

For simplicity it is assumed that both waves have the same frequency. Since the waves are travelling in the same medium they will have the same speed; that is, $v = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$. Thus, $y_{net} = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$. This expression can be expressed as a product of two cosines using the trigonometric identity $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

$$y_{net} = 2A\cos\left[\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{\omega_1+\omega_2}{2}\right)t\right]\cos\left[\left(\frac{k_1-k_2}{2}\right)x - \left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$

As can be seen from this equation, the effect of the interference of two waves of close frequencies is the product of a high frequency wave $\left(\cos\left[\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{\omega_1+\omega_2}{2}\right)t\right]\right)$ and a low frequency wave $\left(\cos\left[\left(\frac{k_1-k_2}{2}\right)x - \left(\frac{\omega_1-\omega_2}{2}\right)t\right]\right)$. The result is travelling wave packets (beats) where the amplitude of the high frequency vibrations are controlled by the low frequency wave. One cycle of of the low frequency wave will result in two beats as a result of the product. That is the beat frequency (f_b) is twice the frequency of the low frequency wave. Since the frequency of the low frequency wave is given by $\frac{|f_1 - f_2|}{2}$, it follows that

$$f_b = |f_1 - f_2|$$

The beat frequency, which is the number of beats per a unit time, is equal to the difference of the frequencies of the interfering waves. And since the beat period (T_b), time taken for one beat, is equal to $T_b = \frac{1}{f_b}$,

$$T_b = \frac{1}{|f_1 - f_2|}$$

Example: Consider the interference of the following two waves: $y_1 = 2\cos(2x - 200t)$ and $y_2 = 2\cos(2.02x - 202t)$.

a) Express the net wave (beat) as a product of two waves.

Solution:

$k_1 = 2$ 1/m; $\omega_1 = 200$ rad/s; $k_2 = 2.02$ 1/m; $\omega_2 = 202$ rad/s; $A = 2$; $y_{net} = ?$

$$y_{net} = 2A\cos\left[\left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\omega_1 + \omega_2}{2}\right)t\right]\cos\left[\left(\frac{k_1 - k_2}{2}\right)x - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right]$$

$$= 2(2)\cos\left[\left(\frac{.02}{2}\right)x + \left(\frac{2}{2}\right)t\right]\cos\left[\left(\frac{4.02}{2}\right)x - \left(\frac{402}{2}\right)t\right]$$

$$= 4\cos[.01x - t]\cos[2.01x - 201t]$$

b) How long does one beat take?

Solution:

$$f_b = |f_1 - f_2| = \left|\frac{\omega_1}{2\pi} - \frac{\omega_2}{2\pi}\right| = \frac{1}{2\pi}|\omega_1 - \omega_2| = \frac{1}{2\pi}|200 - 202| \text{ Hz} = \frac{2}{2\pi} = 0.318 \text{ Hz}$$

$$T_b = \frac{1}{f_b} = \frac{1}{1/\pi} \text{ s} = \pi \text{ s}$$

Example: When a sound of frequency 1000Hz interferes with sound of unknown frequency 2 beats can be heard in 5 seconds.

a) Calculate the two possible frequencies of the unknown wave.

Solution:

$$f_1 = 1000 \text{ Hz}; f_b = \frac{2}{5} \text{ Hz} = 0.4 \text{ Hz}; f_2 = ?$$

$$f_b = |f_1 - f_2|$$

$$f_1 - f_2 = \pm f_b$$

$$f_2 = f_1 \pm f_b = (1000 \pm 0.4) \text{ Hz}$$

b) If both waves have amplitude of 10^{-9} m and the temperature is 20°C, express the beat as a product of two waves (just for one of the unknown frequency).

Solution:

$$\omega_1 = 2\pi f_1 = 2000\pi \text{ rad/s}; \ \omega_2 = 2\pi f_2 = 2000.8\pi \text{ rad/s}; \ A = 10^{-9} \text{ m}; \ T = 293°K; \ y_{net} = ?$$

$$v = 331 \text{ m/s} \sqrt{\frac{T}{273°\text{K}}} = 331\sqrt{\frac{293}{273}} \text{ m/s} = 343 \text{ m/s}$$

$$k_1 = \frac{\omega_1}{v} = \frac{2000\pi}{343} \text{ 1/m} = 18.318 \text{ 1/m}$$

$$k_2 = \frac{\omega_2}{v} = \frac{2000.8\pi}{343} \text{ 1/m} = 18.325 \text{ 1/m}$$

$$y_{net} = 2A\cos\left[\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{\omega_1+\omega_2}{2}\right)t\right]\cos\left[\left(\frac{k_1-k_2}{2}\right)x - \left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$

$$y_net = 2\times10^{-9}\cos(0.004x - 1.26t)\cos(18.32x - 6284t)$$

Practice Quiz 14.2

Choose the best answer

1. Which of the following is not true about standing waves.
   A. The speed of the wave in a string is proportional to the square root of the tension in the string.
   B. For a standing wave of a given length, the greater the number of loops the greater the frequency of the wave.
   C. A standing wave is a wave with equally spaced points of zero vibration.
   D. The length of one loop of a standing wave is equal to half of the wavelength of the wave.
   E. The points of zero vibration of a standing wave are called antinodes.

2. Which of the following is correct about a beat?
   A. The frequency of a beat is equal to the difference between the frequencies of the interfering waves.
   B. A beat is formed by the interference of an oncoming and a reflected wave.
   C. A beat is a wave with points of zero vibrations separated by equal distances.
   D. The period of a beat is equal to the difference between the periods of the interfering waves.
   E. A beat is formed by the interference of two waves of the same frequency.

3. A standing wave is formed when a wave of the form $y = 0.63 \cos(234t - 7.8x)$ m initiated at the origin interferes with its reflection reflected at $x = 10$ m 100% from a denser medium. Calculate the amplitude of the oscillation of a particle located at $x = 9$ m.
   A. 1.258 m
   B. 1.636 m
   C. 0.755 m
   D. 1.132 m
   E. 1.51 m

4. Calculate the wave length of the *3*th harmonic of a standing wave formed in a *4.5* m string clamped at both ends.
   A. *4.2* m
   B. *3* m
   C. *2.7* m
   D. *3.6* m
   E. *3.9* m

5. A string of mass *0.12* kg and length *1.2* m is clamped at both ends. It is subjected to a tension of *110* N. Calculate the frequency of the *6*th harmonic standing wave formed in the string.
   A. *49.749* Hz
   B. *74.624* Hz
   C. *58.041* Hz
   D. *82.916* Hz
   E. *66.332* Hz

6. A sound wave resonance is formed in a pipe of length *5* m open at both ends. Calculate the wavelength of the *11*th harmonic.
   A. *0.636* m
   B. *0.818* m
   C. *0.909* m
   D. *1.091* m
   E. *0.545* m

7. A sound wave resonance is formed in a pipe of length *2* m open at both ends. Calculate the frequency of the *7*th harmonic. (The temperature is *10*°C.)
   A. *353.858* Hz
   B. *766.693* Hz
   C. *648.74* Hz
   D. *589.764* Hz
   E. *707.716* Hz

8. A sound wave resonance is formed in a pipe of length *0.5* m open at one end closed at the other. Calculate the wavelength of the *12*th harmonic.
   A. *0.07* m
   B. *0.087* m
   C. *0.122* m
   D. *0.096* m
   E. *0.061* m

9. A sound wave resonance is formed in a pipe of length *5* m open at one end closed at the other. Calculate the frequency of the 6^{th} harmonic. (The temperature is *35*°C.)
   A. *193.368* Hz
   B. *232.042* Hz
   C. *212.705* Hz
   D. *116.021* Hz
   E. *251.378* Hz

10. A beat is formed by the interference of a wave of frequency *104* Hz and a wave of frequency *109.5* Hz. Calculate the number of beats that can be heard per second.
    A. *5.5* Hz
    B. *100* Hz
    C. *0.182* Hz
    D. *213.5* Hz
    E. *0.005* Hz

11. A beat is formed by the interference of a wave of frequency *1000* Hz and a wave of an unknown frequency. If a beat lasts *0.7* s, which of the following is a possible frequency for the unknown wave?
    A. *600.857* Hz
    B. *1001.429* Hz
    C. *701* Hz
    D. *1101.571* Hz
    E. *1201.714* Hz

12. A beat is formed by the interference of the two sound waves
    $\Delta P_1 = 2 \cos (30x - 10200t)$ Pa
    $\Delta P_1 = 2 \cos (30.294x - 10300t)$ Pa
    Give a formula for the beat (net wave) as function of position and time.
    A. $\Delta P_1 = 4 \cos (56.23x - 37230t) \cos (0.567x - 70t)$ Pa
    B. $\Delta P_1 = 4 \cos (30.147x - 10250t) \cos (0.567x - 70t)$ Pa
    C. $\Delta P_1 = 4 \cos (30.147x - 10250t) \cos (0.147x - 50t)$ Pa
    D. $\Delta P_1 = 4 \cos (23.456x - 20346t) \cos (0.567x - 70t)$ Pa
    E. $\Delta P_1 = 4 \cos (23.456x - 20346t) \cos (0.147x - 50t)$ Pa

15 ELECTROMAGNETIC WAVE

By combining his equations, Maxwell was able to show that whenever a charge is accelerated electrometric waves are propagated. In other words he showed that electric field and the magnetic field satisfy the wave equation. A one dimensional version of the equation can be derived as follows.

At any point in space the electric field and the magnetic field are perpendicular to each other and to the direction of propagation of energy. Let's assume that the direction of the electric field is along the *z*-direction and the magnetic field is along the *y*-direction. Then the direction of the propagation of energy is along the *x*-direction.

Let's apply Faraday's law over the small rectangle of width *dx* and length *dz* on the *x-z* plane. The area of this small rectangle is *dxdz* and thus the magnetic flux crossing it is $\phi_B = Bdxdz$. Therefore $\frac{d\phi_B}{dt} = dxdz\frac{\partial B}{\partial t}$ (It is a partial derivative because it is a constant position process). The line integral $\int \vec{E}\cdot d\vec{s}$ will be zero on the horizontal paths because the electric field and the path are perpendicular to each other. Since the rectangle is small, the electric field can be assumed to be constant on the vertical paths ($E(x)$) on the left and $E(x+dx)$ on the right one. Therefore $\oint \vec{E}\cdot d\vec{s} = E(x+dx)dz - E(x)dz = \frac{\partial E}{\partial x}dxdz$ (It is a partial derivative because it is a constant time process). Now applying Faraday's law $\left(\oint \vec{E}\cdot d\vec{s} = -\frac{d\phi_B}{dt}\right)$, $\frac{\partial E}{\partial x}dxdz = -dxdz\frac{\partial B}{\partial t}$ and

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

Again applying Ampere's law without source (no current) over a small rectangle of sides *dx* and *dy* on the xy-plane, another relationship between the electric and magnetic fields can be obtained. The area of the small rectangle is *dxdy* and the electric flux crossing the rectangle is $\phi_E = Edxdy$. Therefore the displacement current crossing the loop is given by $\varepsilon_0\mu_0\frac{d\phi_E}{dt} = dxdy\varepsilon_0\mu_0\frac{\partial E}{\partial t}$. The line integrals of the magnetic field on the paths parallel to the x-axis are zero because the magnetic field is perpendicular to the paths. On the paths parallel to the y-axis, the magnetic field can be taken to be approximately constant ($B(x)$ on the left side and $B(x+dx)$ on the right side) because the rectangle is infinitely small. Therefore the line integral of the magnetic field on the closed path is given as $\oint B\cdot d\vec{s} = \big(B(x) - B(x+dx)\big)dy = -\frac{\partial B}{\partial x}dxdy$. Now applying Ampere's law without source $\left(\oint \vec{B}\cdot d\vec{s} = \varepsilon_0\mu_0\frac{d\phi_E}{dt}\right)$, $-\frac{\partial B}{\partial x}dxdy = dxdy\varepsilon_0\mu_0\frac{\partial E}{\partial t}$ and

$$\frac{\partial B}{\partial x} = -\varepsilon_0\mu_0\frac{\partial E}{\partial t}$$

Now taking the derivative of this equation with respect to x gives $\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial x}\right) = -\varepsilon_0 \mu_0 \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t}\right) = -\varepsilon_0 \mu_0 \frac{\partial}{\partial x}\frac{\partial E}{\partial t}$. Using the relationship $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$, the following wave equation for the magnetic field is obtained.

$$\frac{\partial^2 B}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$

Comparing with the wave equation $\left(\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\right)$, the magnetic field propagates in space as a wave with speed $v = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$. A similar process can be used to show that the electric field also satisfies the following wave equation.

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$$

Again comparing with the wave equation, the electric field propagates in space as a wave with a speed $v = \sqrt{1/\varepsilon_0 \mu_0}$. Since $\varepsilon_0 = 8.85\times10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$ and $\mu_0 = 4\pi\times10^{-7}\ \frac{\mathrm{mT}}{\mathrm{A}}$, the speed of all electromagnetic waves turned out to be

$$c = \sqrt{1/\varepsilon_0 \mu_0} \approx 3\times10^8 \text{ m/s}$$

which is the known speed of light. This lead to the realization that light is actually a type of electromagnetic wave. Other examples of electromagnetic waves are ultraviolet rays, infrared waves, microwaves, *x*-rays, radio waves and gamma radiation. As shown before, the speed of a wave is related to its wavelength and frequency (or to its wave number and angular frequency) as $v = \lambda f$ or $v = \frac{\omega}{k}$. For electromagnetic waves with $v = c = 3\times10^8$ m/s

$$c = \lambda f = \frac{\omega}{k}$$

Harmonic Electromagnetic Waves

The most common solutions of the wave equation are harmonic waves. The harmonic wave solution for the electric field and magnetic field may be written as

$$E = E_{max}\cos(kx - \omega t)$$

$$B = B_{max}\cos(kx - \omega t)$$

The ratio between these two equations shows that the ratio between the fields is equal to the ratio between their amplitudes.

$$\frac{E}{B} = \frac{E_{max}}{B_{max}}$$

Substituting the harmonic solutions to the relationship between the fields $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$ shows that the ratio of the electric field to the magnetic field is always equal to the speed of light c.

$$\frac{E}{B} = c$$

Example: An electromagnetic wave has a wavelength of 2×10^{-6} m. The amplitude of the electric field is 3×10^{-3} N/C. Give the harmonic wave solution of this electromagnetic wave as a function of position and time.

Solution:

$\lambda = 2\times10^{-6}$ m; $E_{\max} = 3\times10^{-3}$ N/C

$$E = E_{max}\cos(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\times10^{-6}} = \pi\times10^{6}\ \text{1/m}$$

$$\omega = kc = \pi \times 10^6 \times 3 \times 10^8 = 3\pi \times 10^{14} \text{ rad/s}$$

$$E = \left(3 \times 10^{-3} \text{ N/C}\right) \cos\left\{\left(\pi \times 10^3\right)x - \left(3\pi \times 10^{14}\right)t\right\}$$

$$B = B_{max} \cos\left(kx - \omega t\right)$$

$$B_{max} = \frac{E_{max}}{C} = \frac{3 \times 10^{-3}}{3 \times 10^8} \text{ T} = 10^{-11} \text{ T}$$

$$B = \left(10^{-11} \text{ T}\right) \cos\left\{\left(\pi \times 10^3\right)x - \left(3\pi \times 10^{11}\right)t\right\}$$

Energy density of an electromagnetic wave

The energy density of an electromagnetic wave is the sum of the energy densities due to its electric field and magnetic field. As shown in previous chapters, the energy densities due to electric field and magnetic field are respectively given by $u_E = \frac{1}{2}\varepsilon_0 E^2$ and $u_B = \frac{1}{2u_0} B^2$. Therefore the electromagnetic energy density u is given by $u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2u_0} B^2$. With $B = \frac{E}{c} = \sqrt{\varepsilon_0 \mu_0} E$, this simplifies to

$$u = \varepsilon_0 E^2$$

Also, substituting $E = \sqrt{\varepsilon_0 \mu_0}\ B$, u can be expressed in terms of B as

$$u = \frac{B^2}{\mu_0}$$

This is instantaneous energy density. The average of the energy density $\left(\overline{u}\right)$ for a harmonic wave can be obtained by integrating the energy density with time over one cycle and dividing by its period.

$$\overline{u} = \frac{1}{2}\varepsilon_0 E_{max}^2 \quad \text{or} \quad \overline{u} = \frac{1}{2\mu_0} B_{max}^2$$

Poynting Vector

Poynting Vector $\left(\vec{S}\right)$ is a vector that represents the amount of electromagnetic energy that crosses a unit perpendicular area per a unit time. Its direction is the direction of the propagation of energy which is perpendicular to both the electric and magnetic field. Suppose electromagnetic energy contained in a small cylinder of length *dx* and cross-sectional area $dA_\perp$ crosses the cross-section of the cylinder in a time interval *dt*. This amount of energy is equal to $udxdA_\perp$. Therefore the magnitude of the poynting vector is $S = \frac{udxdA_\perp}{dA_\perp dt} = u\frac{dx}{dt}$. But $\frac{dx}{dt} = c$. Therefore the magnitude of the poynting vector is equal to the product of the speed of light and the electromagnetic energy density.

$$S = cu$$

Using the expressions for the energy density in terms of the electric field or magnetic field, the magnitude of the poynting vector may also be expressed $S = \frac{E^2}{\mu_0 C} = \frac{C}{\mu_0}B^2$. The average of the poynting depends on the amplitudes as $\bar{S} = c\bar{u} = \frac{1}{2}\frac{E_{max}^2}{\mu_0 C} = \frac{1}{2}\frac{C}{\mu_0}B_{max}^2$.

Since the poynting vector is perpendicular to both the electric and magnetic fields, a unit vector $\left(\hat{e}_s\right)$ in the direction of the poynting vector can be written as $\hat{e}_s = \frac{\vec{E}\times\vec{B}}{\left|\vec{E}\times\vec{B}\right|}$. But $\left|\vec{E}\times\vec{B}\right| = EB$ because the fields are perpendicular to each other. Therefore $\vec{S} = S\hat{e}_s = \frac{cu}{EB}\vec{E}\times\vec{B}$. Substituting the expressions for the energy density and the magnetic field in terms of the electric field, it can be shown that $\frac{cu}{EB} = \frac{1}{\mu_0}$.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

Example: The frequency of a certain electromagnetic wave is 4×10^{10} *Hz. The amplitude of its electric field is* 4×10^{-6} *N/m. Assuming the wave is harmonic.*

a) Calculate the average energy density for this radiation.

Solution:

$f = 4\times10^{10}$ Hz; $E_{\max} = 4\times10^{-6}$ N/C; $\bar{u} = ?$

$$\bar{u} = \frac{\varepsilon_0 E_{max}^2}{2} = \frac{\left(8.85\times10^{-12}\right)\left(4\times10^{-6}\right)^2}{2} \text{ J/m}^3 = 7.08\times10^{-23} \text{ J/m}^3$$

b) Calculate the average of electromagnetic energy that crosses a unit perpendicular are per a unit time.

Solution:

$\bar{S} = ?$

$$\bar{s} = c\bar{u} = \left(3\times10^{8}\right)\left(7.08\times10^{-23}\right) \text{ W/m}^2 = 2.1\times10^{-15} \text{ W/m}^2$$

Practice Quiz 15

Choose the best answer

1. Which of the following statements is not a correct statement?
   A. All electromagnetic waves have the same speed in vacuum (air).
   B. Electromagnetic wave is a transverse wave.
   C. In an electromagnetic wave, the electric field and the magnetic field are always perpendicular to each other.
   D. The ratio between the electric field and magnetic field of an electromagnetic wave is proportional to the wavelength of the electromagnetic wave
   E. The electric and magnetic fields of an electromagnetic wave are always perpendicular to the direction of propagation of energy

2. Calculate the wavelength of an electromagnetic wave whose frequency is *4.7e14* m.
   A. *11.475e-7* m
   B. *9.924e-7* m
   C. *6.383e-7* m
   D. *7.418e-7* m
   E. *5.613e-7* m

3. For a certain harmonic electromagnetic wave of frequency *6.4e15* Hz, the amplitude of the electric field is *1.4e-3* N/C. The electric field may be expressed as a function of position and time as
   A. *1.4e-3* N / C *cos (67.12e15t – 90.082e6x)*
   B. *1.4e-3* N / C *cos (40.212e15t – 134.041e6x)*
   C. *1.4e-3* N / C *cos (67.12e15t – 134.041e6x)*
   D. *1.4e-3* N / C *cos (40.212e15t – 155.964e6x)*
   E. *1.4e-3* N / C *cos (44.234e15t – 155.964e6x)*

4. For a certain harmonic electromagnetic wave of frequency *8.2e15* Hz, the amplitude of the electric field is *7.4e-3* N / C. The magnetic field may be expressed as a function of position and time as
   A. *20.743e-12* T *cos (91.948e15t – 273.9e6x)*
   B. *24.667e-12* T *cos (91.948e15t – 77.079e6x)*
   C. *24.667e-12* T *cos (51.522e15t – 171.74e6x)*
   D. *11.071e-12* T *cos (51.522e15t – 273.9e6x)*
   E. *20.743e-12* T *cos (23.124e15t – 171.74e6x)*

5. For a certain harmonic electromagnetic wave, the maximum value of the electromagnetic energy is *5.1e-16* J. Calculate the amplitude of the electric field.
   A. *2.387e-3* N / C
   B. *3.562e-3* N / C
   C. *9.654e-3* N / C
   D. *13.212e-3* N / C
   E. *7.595e-3* N / C

6. For a certain harmonic electromagnetic wave, the average value of the electromagnetic energy is *6.9e-16* J. Calculate the amplitude of the magnetic field.
   A. *7.845e-11* T
   B. *2.432e-11* T
   C. *4.164e-11* T
   D. *0.805e-11* T
   E. *4.744e-11* T

7. For a certain harmonic electromagnetic wave, the amplitude of the magnetic field is *1.4e-11* T. Calculate the maximum possible value of the electromagnetic energy that crosses a unit perpendicular area per a unit time.
   A. *1.032e-8* W / m^2
   B. *8.683e-8* W / m^2
   C. *4.679e-8* W / m^2
   D. *5.487e-8* W / m^2
   E. *3.619e-8* W / m^2

8. For a certain harmonic electromagnetic wave, the amplitude of the electric field is *4.8e-3* N / C. Calculate the average value of the electromagnetic energy that crosses a unit perpendicular area per a unit time.
   A. *3.457e-8* W / m^2
   B. *4.029e-8* W / m^2
   C. *3.056e-8* W / m^2
   D. *4.357e-8* W / m^2
   E. *1.375e-8* W / m^2

9. For a certain harmonic electromagnetic wave, the average electromagnetic energy density is *9.1e-16* J. Calculate the average electromagnetic energy that crosses a unit perpendicular area per a unit time.
   A. *27.3e-8* $\mathrm{W/m^2}$
   B. *51.198e-8* $\mathrm{W/m^2}$
   C. *42.768e-8* $\mathrm{W/m^2}$
   D. *15.67e-8* $\mathrm{W/m^2}$
   E. *47.815e-8* $\mathrm{W/m^2}$

10. For a certain electromagnetic wave, the instantaneous electric and magnetic fields at a certain location are *4.8e-3* N/C $\boldsymbol{i}$ and *9.1e-11* T $\boldsymbol{j}$ respectively. Calculate the corresponding poynting vector.
    A. *9.369e-8* $\mathrm{W/m^2}$ $\boldsymbol{i}$
    B. *43.357e-8* $\mathrm{W/m^2}$ $\boldsymbol{j}$
    C. *43.357e-8* T $\boldsymbol{k}$
    D. *34.759e-8* $\mathrm{W/m^2}$ $\boldsymbol{j}$
    E. *34.759e-8* $\mathrm{W/m^2}$ $\boldsymbol{k}$

16 LIGHT AND OPTICS

Your goal for this chapter is to learn about reflection, refraction, total internal reflection and dispersion of light.

The History of Light

In the seventeenth century, Isaac Newton stated that light is made up of corpuscles (particles). But later it was found out that light displays wave properties, and Christian Huygens stated that light is a wave. In the mid nineteenth century, Maxwell developed the four equations of electricity and magnetism known as Maxwell's equations. Based on these equations, Maxwell showed that light is a small subset of a large group of waves called electromagnetic waves. This established the wave nature of light firmly. But in the late nineteenth century and early twentieth century, some experiments that contradicted classical physics were done. These were the black body radiation experiment, the photoelectric experiment and the Michelson-Morley experiment. Max Planck found out that the findings of the black body radiation can be explained only if he assumes that light is propagated in the form of particles. His assumptions were confirmed by subsequent experiments mainly the photoelectric experiment. The current understanding of light is that light is both a particle and wave. This means in some experiments it behaves like a wave and in some experiments like a particle. This is called the dual property of light.

The Wave Nature of Light

The wave nature of light is best described by means of Maxwell's equations. Maxwell's equations are four equations that encompass the vast experimental findings of electricity and magnetism. The first equation is a representation of Coulomb's law which deals with the force between charged objects. The second equation is a mathematical representation of the fact that magnetic field lines form complete loops. The third equation is a representation of Ampere's law modified by theoretical prediction of Maxwell. Ampere's Law states that current gives rise to magnetic field and Maxwell's theoretical prediction states a time varying electric field produces magnetic field. The fourth equation is a representation of Faraday's law which states that a time varying magnetic field produces electric field or induced emf.

By combining these equations, Maxwell was able to predict that whenever a charge is accelerated, electromagnetic wave is propagated. This theoretical prediction was tested for the first time by the German scientist by the name Hertz (the unit of frequency was named Hertz in honor of this scientist) who produced electromagnetic waves by accelerating charges. This was put into practical application for the first time by the Italian inventor Marconi who invented the radio. Today electromagnetic waves are a part of our daily life. Radio signals, television signals, cellphone signals and others are carried by electromagnetic waves.

All electromagnetic waves have the same speed which is the speed of light. The speed of light (c) in vacuum is $3e8$ m/s.

$$c = 3e8 \text{ m/s}$$

Electromagnetic waves are classified according to their wavelengths. The range of wavelengths that produce sensation of vision are called light waves. The range of wavelengths that produce sensation of heat are called infrared waves. The range of wavelengths that are used to carry radio signals are called radio waves. Other examples include x-rays (used in medicine to obtain pictures of internal body parts), ultraviolet rays (which produces vitamin D in our bodies) and Gamma radiation (which is the most energetic radiation).

The physical quantities that vary as a function of position and time for electromagnetic waves are electric field and magnetic field. The electric field and the magnetic field are perpendicular to each other and to the direction of propagation of energy. Since electromagnetic waves are waves, they satisfy the wave equation.

$$c = f\lambda$$

Where f is frequency and λ is wavelength.

Example: Violet light has a wavelength of 400 nm. Calculate its frequency.

Solution: $\lambda = 400$ nm $= 400e\text{-}9$ m $= 4e\text{-}7$ m.; $f = ?$

$$f = c/\lambda = 3e8/4e\text{-}7 \text{ Hz} = 7.5e14 \text{ Hz}$$

Particle Nature of Light

The black body radiation experiment is an experiment that studied the intensity distribution of the different wavelengths emitted by a hot object. A graph of intensity as a function of wavelength was obtained. Classical physics failed to explain the results of this experiment. Max Planck has to state the following two postulates to explain the experiment:

1. Light is propagated in the form of particles called photons.
2. The energy of a photon is directly proportional to the frequency of light.

If E is the energy of a photon and f is the frequency of the light, then

$$E = hf$$

h is a universal constant called Planck's constant. Its value is *6.6e-34* Js.

$$h = 6.6e\text{-}34 \text{ Js}$$

For light containing a number of photons, the total energy is obtained by multiplying the energy of one photon by the number of photons.

$$E_n = nhf$$

Where n is the number of photons and E_n is the energy of n photons. Max Planck's postulates were confirmed by the findings of the photoelectric experiment which was explained by Albert Einstein. These postulate gave rise to the formation of a new branch of physics called quantum mechanics. Quantum mechanics has been used successfully at the atomic level.

Example: Calculate the energy of *5000* photons of red light. The wavelength of red light is *700* nm.

Solution: $n = 5000$; $\lambda = 700$ nm $= 7e\text{-}7$ m ($f = c/\lambda$); $E_n = ?$

$$f = c/\lambda = 3e8/7e\text{-}7 \text{ Hz} = 4.3e14 \text{ Hz}$$

$$E_n = nhf$$

$$E_{5000} = 5000 * 6.6e\text{-}34 * 4.3e14 \text{ J} = 1.3e\text{-}15 \text{ J}$$

Reflection of Light

Reflection of light is the bouncing of light from a surface. The light ray that hits the surface is called incident ray. The light ray reflected from the surface is called the reflected ray. The line perpendicular to the surface at the point of impact is called the normal line. The angle formed between the incident ray and the normal line is called the angle of incidence. The angle formed between the reflected ray and the normal line is called the angle of reflection. The **law of reflection** states that the angle of incidence (θ_i) and the angle of reflection (θ_r) are equal.

$$\theta_i = \theta_r$$

Example: A light ray is incident on a flat mirror. The angle formed between the surface of the mirror and the incident ray is *25°*. Calculate the angle of reflection.

Solution: Since the angle of incidence is angle formed with the normal and the incident ray makes an angle of $25°$ with the surface, the angle of incidence is $90° - 25° = 65°$.

$\theta_i = 65°$; $\theta_r = ?$

$$\theta_r = \theta_i = 65°$$

Example: Two mirrors are connected at an angle of $120°$. A light ray is incident on one of the mirrors at an angle of incidence of $70°$. Calculate its angle of reflection on the second mirror.

Solution: Since the angle of incidence on the first mirror is $70°$, the angle of reflection on the first mirror is $70°$. The angle formed between the surface of the first mirror and the reflected ray is $90° - 70° = 20°$. The reflected ray will continue to hit the second mirror and reflected. The two surfaces of the mirror and the path of the light ray from the first to the second mirror form a triangle. The angle formed between the second mirror and the light ray incident on the second mirror is $180° - (120° + 20°) = 40°$. The angle of incidence on the second mirror is $90° - 40° = 50°$. Therefore the angle of reflection on the second mirror is $50°$.

Refraction

Refraction is the bending of light as light crosses the boundary between two mediums. The light ray incident on the boundary is called incident ray. The ray past the boundary is called the refracted ray. The line that is perpendicular to the boundary at the point of impact is called the normal line. The angle formed between the incident ray and the normal line is called the angle of incidence. The angle formed between the refracted ray and the normal line is called the angle of refraction.

As light enters a medium from a vacuum (or approximately air), the speed and the wavelength of the light decrease while the frequency remains the same. The ratio between the speed of light (c) in vacuum and the speed of light (v) in a medium is called the *refractive index* (n) of the medium.

$$n = c/v$$

Refractive index is unit-less. From this definition of refractive index, it is clear that the refractive index (n_a) of vacuum (air) is one and the refractive index of any other medium is greater than one. The refractive index of water (n_w) and glass (n_g) are $4/3$ and $3/2$ respectively. If the wavelength of light is λ_v in vacuum and λ_m in a medium, since frequency remains the same $n = c/v = f\lambda_v/(f\lambda_m)$ and

$$n = \lambda_v/\lambda_m$$

Example: Calculate the speed of light in glass.

Solution: $n_g = 1.5$; $\mathrm{v_g}$ = ?

$$v_g = c/n_g = 3e8/1.5 \text{ m/s} = 2e8 \text{ m/s}$$

Example: The wavelength of violet light in vacuum is 400 nm. Calculate its wavelength in water.

Solution: $\lambda_v = 400$ nm $= 4e\text{-}7$ m; $n_w = 4/3$; $\lambda_w = ?$

$$\lambda_w = \lambda_v/n_w = 4e\text{-}7/(4/3) \text{ m} = 3e\text{-}7 \text{ m}$$

Snell's law (law of refraction) states that the ratio between the sine of the angle of incidence and the sine of the angle of refraction is equal to the ratio between the speeds of light in the respective mediums. If light enters medium *2* from medium *1* at an angle of incidence θ_1 and the angle of refraction in medium *2* is θ_2, then $\sin(\theta_1)/\sin(\theta_2) = v_1/v_2 = (c/n_1)/(c/n_2) = n_2/n_1$. Therefore, Snell's can be mathematically expresses as

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

It can be easily deduced from this equation that light bends towards the normal as it enters an optically denser (higher refractive index) medium and bends away from the normal as it enters an optically less dense medium. A light ray perpendicular to the boundary passes straight unbent.

Example: A light ray enters water (from air) at an angle of incidence $56°$. Calculate the angle of refraction in water.

Solution: $n_a = 1$; $n_w = 4/3$; $\theta_a = 56°$; $\theta_w = ?$

$$n_a \sin(\theta_a) = n_w \sin(\theta_w)$$

$$\sin(\theta_w) = n_a \sin(\theta_a)/n_w = 1 * \sin(56°)/(4/3) = 0.62$$

$$\theta_w = \arcsin(0.62) = 38.3°$$

Example: A light ray enters water (from air) on glass at an angle of incidence of $65°$. Calculate the refraction angle in glass. (The air-water boundary and the water-glass boundary are parallel)

Solution: First the angle of refraction in water should be calculated from the air-water boundary. The angle of refraction in water and the angle of incidence on the water-glass boundary are equal because they are alternate interior angles. Then the angle of refraction in glass can be obtained from the water-glass boundary.

$n_a = 1$; $n_w = 4/3$; $n_g = 1.5$; $\theta_a = 65°$; $\theta_g = ?$

$$n_a \sin(\theta_a) = n_w \sin(\theta_w)$$

$$\sin(\theta_w) = n_a \sin(\theta_a)/n_w = 1 * \sin(65°)/(4/3) = 0.7$$

$$\theta_w = \arcsin(0.7) = 44.4°$$

$$n_w \sin(\theta_w) = n_g \sin(\theta_g)$$

$$\sin(\theta_g) = n_w \sin(\theta_w)/n_g = (4/3) * \sin(44.4°)/(1.5) = 0.5$$

$$\theta_g = \arcsin(0.5) = 30°$$

Practice Quiz 16.1

Choose the best answer

1. The scientist who first stated that light is made up of corpuscles is
   A. Huygens
   B. Hertz
   C. Newton
   D. Planck
   E. Maxwell

2. Which of the following is a correct statement?
   A. The electric and magnetic fields of an electromagnetic wave are parallel to the direction of propagation of energy.
   B. For electromagnetic waves, the physical quantities that vary as a function of position and time are electric and magnetic fields.
   C. Different electromagnetic waves have different speeds in vacuum.
   D. According to the current understanding of light, light is a wave.
   E. The electric and magnetic fields of an electromagnetic wave are parallel to each other.

3. Which of the following is a correct statement?
   A. As light enters a denser medium, its wavelength increases.
   B. As light enters an optically less dense medium, it bends towards the normal.
   C. Refractive index of any medium is less or equal to one.
   D. As light enters a denser medium its frequency remains the same.
   E. As light enters an optically denser medium, its speed increases

4. Which of the following is a correct statement?
   A. Refractive index of a medium is equal to the ratio of the wavelength of light in vacuum to the wavelength of the light in the medium.
   B. Refraction is the bending of light as light hits an obstacle.
   C. Refractive index of a medium is equal to the ratio of the speed of light in the medium to the speed of light in vacuum.
   D. Angle of reflection may or may not be equal to angle of incidence.
   E. As light enters medium *2* from medium *1*, the ratio of the sine of the angle of incidence in medium *1* to the refractive index of medium *1* is equal to the ratio of the sine of the angle of refraction in medium *2* to the refractive index of medium *2*.

5. One photon of a certain light has an energy of *3.8e-19* J. Calculate the wavelength of the light.
   A. *364.737e-9* m
   B. *312.632e-9* m
   C. *573.158e-9* m
   D. *521.053e-9* m
   E. *416.842e-9* m

6. Calculate the energy of *100* photons of light of wavelength *6.1e-7* m.
   A. *45.443e-18* J
   B. *32.459e-18* J
   C. *35.705e-18* J
   D. *22.721e-18* J
   E. *38.951e-18* J

7. Two mirrors are connected so that the angle formed between them is *105*°. A light ray is incident on one of the mirrors making an angle of *14*° with the surface of the mirror in such a way that the reflected ray is incident on the second mirror. Calculate the angle formed between the light ray reflected from the second mirror and the surface of the second mirror.
   A. *61*°
   B. *56*°
   C. *58*°
   D. *60*°
   E. *63*°

8. Two mirrors are joined at an angle. A light ray incident on one of the mirrors making an angle of *29*° with the surface of the mirror is reflected from the second mirror making an angle of *40*° with the surface of the mirror. Calculate the angle formed between the two mirrors.
   A. *113*°
   B. *111*°
   C. *115*°
   D. *108*°
   E. *110*°

9. The speed of light in a certain medium is *2.1e8* m/s. Calculate the refractive index of the medium.
   A. *1.571*
   B. *1.714*
   C. *2*
   D. *1.429*
   E. *1.857*

10. A certain light has a wavelength of *4.9e-7* m in vacuum. Calculate its wavelength in a medium whose refractive index is *1.63*.
    A. *4.209e-7* m
    B. *3.307e-7* m
    C. *3.607e-7* m
    D. *3.006e-7* m
    E. *3.908e-7* m

11. A light ray enters glass from air at an angle of incidence of *50*°. Calculate the angle of refraction in glass. Refractive index of water is *1.5*.
    A. *30.71°*
    B. *36.852°*
    C. *39.923°*
    D. *33.781°*
    E. *18.426°*

12. Light enters a certain medium from air at an angle of incidence of *50*°. If the angle of refraction in the medium is *35*°, calculate the refractive index of the medium.
    A. *1.469*
    B. *1.336*
    C. *0.801*
    D. *1.068*
    E. *1.736*

13. Light enters a certain medium from air at an angle of incidence of *20*°. If the speed of light in the medium is *2.7e8* m / s, calculate the angle of refraction in the medium.
    A. *17.928*°
    B. *16.135*°
    C. *23.306*°
    D. *10.757*°
    E. *21.513*°

14. Water is placed on top of glass. A light ray enters the water from air at an angle of incidence of *55*° Calculate the angle of refraction in glass. Refractive index of water and glass are *4 / 3* and *1.5* respectively.
    A. *43.03*°
    B. *33.1*°
    C. *29.79*°
    D. *26.48*°
    E. *19.86*°

Dispersion of Light

Dispersion is the separation of white light into different colors as light enters a medium from air. White light is composed of seven different colors. Arranged in increasing order of wavelength, these are violet, indigo, blue, green, yellow, orange and red (Abbreviated as VIBGYOR). Violet has the shortest wavelength and red has the longest wavelength. The reason white separates into its component colors as light enters a medium is because the refractive index of a medium depends on the wavelength of the light. The refractive index increases as the wavelength decreases. That is, violet has the largest refractive index and red has the smallest refractive index. According to Snell's law, the greater the refractive index the smaller the angle of refraction or the greater the deviation angle from the path of the incident light. Thus, as light enters a medium from air, violet light will be bent by the largest angle and red light will be bent by the smallest angle.

A good example of dispersion is the rainbow. Rainbow happens because the cloud has different refractive indexes for the different colors of light.

Example: White light enters glass (from air) at an angle of incidence of *65°*. The refractive indices of the glass for violet and red light are respectively *1.52* and *1.48*. Calculate the angle formed between red light and violet light after refraction.

Solution: $\theta_a = 65°$; $n_a = 1$; $n_{gr} = 1.48$; $n_{gv} = 1.52$; $\theta = \theta_{gr} - \theta_{gv} = ?$

$$n_{nr} \sin(\theta_{gr}) = n_a \sin(\theta_a)$$

$$\sin(\theta_{gr}) = n_a \sin(\theta_a)/n_{gr} = 1 * \sin(65°)/1.48 = 0.612$$

$$\theta_{gr} = \arcsin(0.612) = 37.8°$$

$$n_{nv} \sin(\theta_{gv}) = n_a \sin(\theta_a)$$

$$\sin(\theta_{gv}) = n_a \sin(\theta_a)/n_{gv} = 1 * \sin(65°)/1.52 = 0.596$$

$$\theta_{gv} = \arcsin(0.596) = 36.6°$$

$$\theta = \theta_{gr} - \theta_{gv} = 37.8° - 36.6° = 1.2°$$

Total Internal Reflection

As light enters an optically less dense medium, it bends away from the normal. As the angle of incidence in the denser medium is increased, for a certain angle the angle of refraction will be $90°$; that is, the light ray will be refracted parallel to the boundary. The angle of incidence in the denser medium for which the angle of refraction is $90°$ is called the *critical angle* (θ_c) of the boundary. If the refractive indexes of the less dense and more dense medium are $n_<$ and $n_>$ respectively, then $n_> \sin(\theta_c) = n_< \sin(90°) = n_<$; and the critical angle of the boundary is given as follows.

$$\theta_c = \arcsin(n_</n_>)$$

For angles of incidence less than the critical angle, both reflection and refraction take place. The fact that we can see our face in water indicates that some of the light rays are reflected; and the fact that we can see objects inside water indicates that some of the light rays are refracted. But for angles greater than the critical angle, only reflection takes place. *Total internal reflection* is a phenomenon where only reflection takes place at the boundary between two mediums and occurs only when light enters a less dense medium at an angle of incidence greater than the critical angle.

Example: Calculate the critical angle for the boundary formed between

a) air and water.

Solution: $n_< = n_a = 1$; $n_> = n_w = 4/3$; $\theta_c = ?$

$$\theta_c = arcsin\ (n_</n_>) = 48.6°$$

b) water and glass.

Solution: $n_< = n_w = 4/3$; $n_> = n_g = 1.5$; $\theta_c = ?$

$$\theta_c = arcsin\ (n_</n_>) = 62.8°$$

Example: A light source is placed in water at a depth of 0.5 m. Calculate the radius of a circular region at the surface of the water from which light rays come out.

Solution: Light rays from the light source will hit the air-water boundary at different angles of incidence. Only the light rays whose angle of incidence is less than the critical angle can be refracted into air. The light rays whose angle of incidence is greater than the critical angle will be reflected back because total internal reflection takes place. Because of this, light rays will come out of a certain circular region of the surface only. The angle of incidence for the light rays that fall on the boundary of this circular region is equal to the critical angle of the boundary. The radius can be calculated from the right angled triangle formed by a line connecting the source with the center of the circle (*a*), the line that connects the center of the circle to a point on the boundary of the circular region (*r*) and the line connecting the source to the point on the boundary. The first two lines are perpendicular to each other and the angle formed between the last two lines is the critical angle. Thus $\theta_c = arctan\ (r/a)$.

$n_< = n_a = 1$; $n_> = n_w = 4/3$; $a = 0.5$ m; $r = ?$

$$\theta_c = arcsin\ (n_</n_>) = arcsin\ (1/(4/3)) = 48.6°$$

$$tan\ (\theta_c) = r/a$$

$$r = a\ tan\ (\theta_c) = 0.5 * tan\ (48.6°) = 0.57 \text{ m}$$

Example A light ray is incident on one of the legs of a $45°$ right angled glass prism perpendicularly. Trace the path of the light ray.

Solution: Since the light ray is perpendicular to the surface (angle of incidence zero), it will enter undeflected. Then it will be incident on the glass-air boundary on the larger face (hypotenuse). Since it is a $45°$ prism, from simple geometry, it can be shown that the angle of incidence on this boundary is $45°$. Since the light ray is incident on the denser medium (glass), what happens depends on the critical angle of the boundary. If the critical angle is greater than $45°$ it can be refracted. But if the critical angle is less than $45°$, total internal reflection will take place and the light ray will be incident on the other leg of the prism perpendicularly (as can be shown by simple geometry) and will be refracted to air undeflected.

$n_< = n_a = 1;\ n_> = n_g = 1.5;\ \theta_c = ?$

$$\theta_c = \arcsin\,(n_</n_>) = \arcsin\,(1/1.5) = 41.8°$$

Since the critical angle is less than the angle of incidence at the hypotenuse, total internal reflection takes place and the light ray is incident on the other leg perpendicularly and is reflected to air undeflected.

Practice Quiz 16.2

Choose the best answer

1. Which of the following is a correct statement?
   A. Dispersion is the separation of white light into different colors as light enters a medium from vacuum (air).
   B. All colors of light have the same refractive index in a given medium.
   C. As wavelength of light increases, refractive index increases.
   D. The seven different colors of light listed in decreasing order of wavelength are violet, indigo, blue, green, yellow, orange and red.
   E. Dispersion occurs because different colors of light have different speeds in vacuum.

2. The color of light with the smallest refractive index is
   A. violet
   B. red
   C. green
   D. yellow
   E. blue

3. Which of the following is a correct statement?
   A. When light enters a less dense medium at an angle of incidence greater than the critical angle, both reflection and refraction take place.
   B. When light enters a less dense medium both reflection and refraction take place for all angles of incidence.
   C. When light enters a denser medium, both reflection and refraction takes place.
   D. Total internal reflection cannot occur when light enters a less dense medium.
   E. When light enters a less dense medium at an angle of incidence less than the critical angle, only refraction takes place.

4. Calculate the critical angle for the boundary between two mediums of refractive indexes *2.1* and *1.2*.
   A. $31.365°$
   B. $34.85°$
   C. $20.91°$
   D. $41.82°$
   E. $24.395°$

5. When light enters a medium of refractive index *1.3* from a medium of refractive index *2*, for which of the following angle of incidence would both reflection and refraction take place?
   A. *42.674°*
   B. *43.665°*
   C. *37.019°*
   D. *44.776°*
   E. *41.977°*

6. A source of light is placed *0.6* m below the surface of water in a pond. Because of total internal reflection, light come out of the surface only from a certain circular region. Calculate the radius of this circular region. (Refractive index of water is *4/3*).
   A. *0.884* m
   B. *0.816* m
   C. *0.68* m
   D. *0.408* m
   E. *0.612* m

7. The refractive indexes of red light and violet light in a certain glass are *1.48* and *1.52* respectively. If white light enters this glass from air at angle of incidence of *50°*, the angle of refractions for violet and red light respectively are
   A. *30.263°, 37.405°*
   B. *30.263°, 31.171°*
   C. *18.158°, 40.523°*
   D. *24.211°, 37.405°*
   E. *24.211°, 31.171°*

8. The refractive indexes of red light and violet light in a certain glass are *1.48* and *1.52* respectively. If white light enters this glass from air at angle of incidence of *30°*, the angle formed between the violet light and red light after refraction is
   A. *0.594°*
   B. *0.54°*
   C. *0.756°*
   D. *0.432°*
   E. *0.702°*

17 MIRRORS AND LENSES

Your goal for this chapter is to learn about the properties of images formed by mirrors and lenses.

Your goals for this chapter are to learn about the properties of the images formed by mirrors and lenses.

An image of a point formed by a mirror is the point at which light rays from the point converge or seem to converge after reflection. An image of a point formed by a lens (a piece of glass with spherical surfaces) is the point at which light rays from the point converge or seem to converge after refraction. There are two kinds of images: real and virtual. A *real image* is an image where actual light rays converge. A real image can be captured in a screen. An example is an image formed by a cinema projector. A *virtual image* is an image where actual light rays do not converge but seem to converge. A virtual image cannot be captured in a screen. An example is an image formed by a flat mirror.

Mirrors

Flat Mirrors

The following diagram shows image formation by a flat mirror.

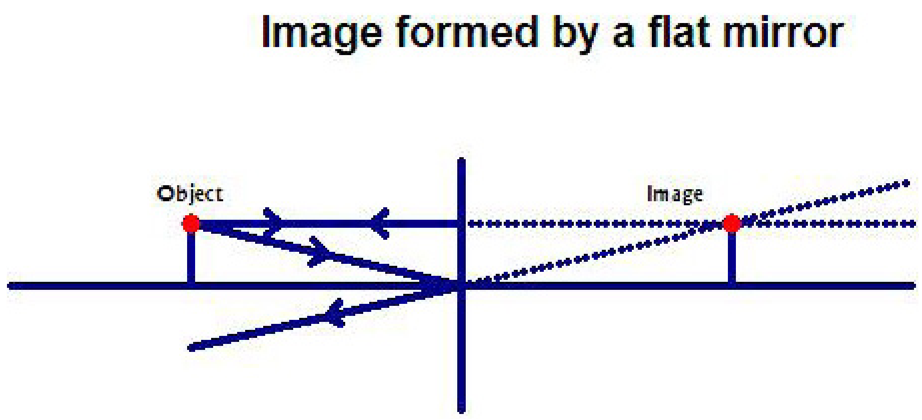

Figure 17.1

The image formed by a flat mirror has the following properties.

1. It is a virtual image.
2. It is located behind the mirror.
3. It has the same size as the object. Its perpendicular distance from the mirror is equal to the perpendicular distance of the object from the mirror.
4. It is erect (not inverted) in a direction parallel to the mirror.
5. It is laterally inverted. In other words the image is inverted in a direction perpendicular to the mirror. For example the image of an arrow pointing towards the mirror is an arrow pointing towards the arrow itself.

Concave mirror

A concave mirror is a spherical mirror with the reflecting surface being the inner surface. The following diagram shows a concave mirror.

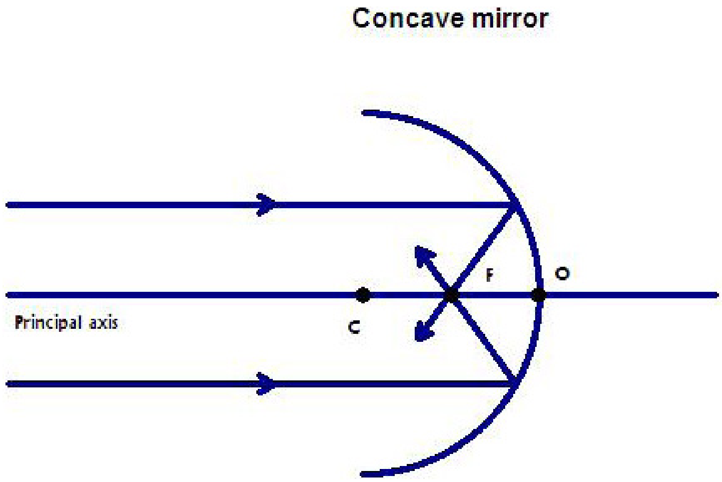

Figure 17.2

The center of the spherical surface is called the *center of curvature* (point C in the diagram) of the mirror. The midpoint of the mirror is called the *center of the mirror* (point O in the diagram). The line joining the center of curvature and the center of the mirror is called the *principal axis* of the mirror. The point at which light rays parallel to the principal axis converge after reflection is called the *focus* (point F in the diagram) of the mirror. The focus of a concave mirror is real because actual light rays meet at the point. The focus is located midway between the center of curvature and the center of the mirror. The distance between the focus and the center of the mirror is called the *focal length* (f) of the mirror.

Only two light rays originating from a point are needed to construct its image. The image is the point at which these light rays converge or seem to converge after reflection. There are 3 special light rays that can be used when constructing an image.

1. A light ray parallel to the principal axis is reflected through the focus.
2. A light ray through the focus is reflected parallel to the principal axis.
3. A light ray through the center of curvature returns in its own path.

The following diagram shows the construction of the image formed by a concave mirror when the object is placed beyond the center of curvature.

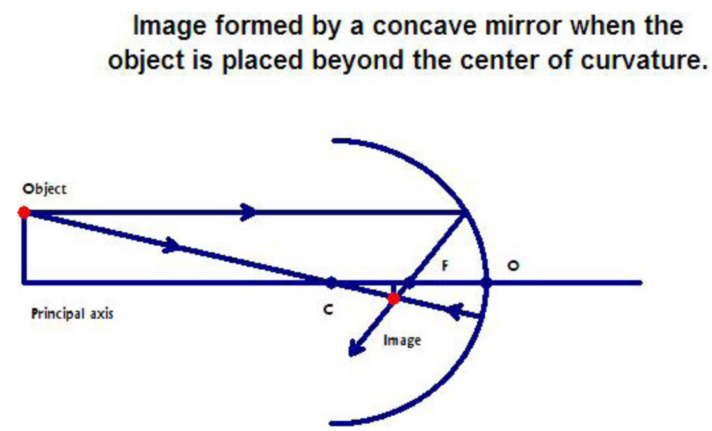

Figure 17.3

The image formed by a concave mirror when the object is located beyond the center of curvature has the following properties.

1. The image is real.
2. The image is inverted.
3. The image is diminished.
4. The image is located between the focus and the center of curvature on the same side as the object.

The following diagram shows the construction of the image formed by a concave mirror when the object is located between the center of curvature and the focus.

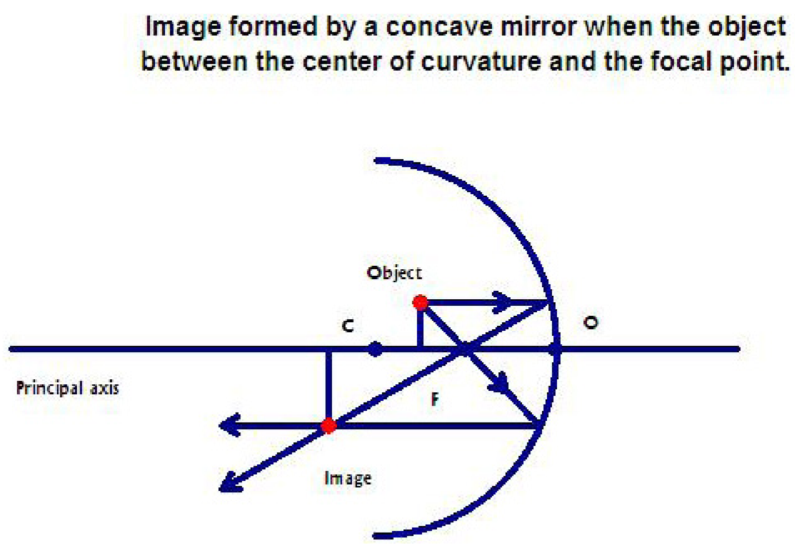

Figure 17.4

An image formed by a concave mirror when the object is placed between the center of curvature and the focus has the following properties.

1. The image is real.
2. The image is inverted.
3. The image is enlarged.
4. The image is located beyond the center of curvature on the same side as the object.

The following diagram shows the image construction of the image formed by a concave mirror when the object is located between the focus and the center of the mirror.

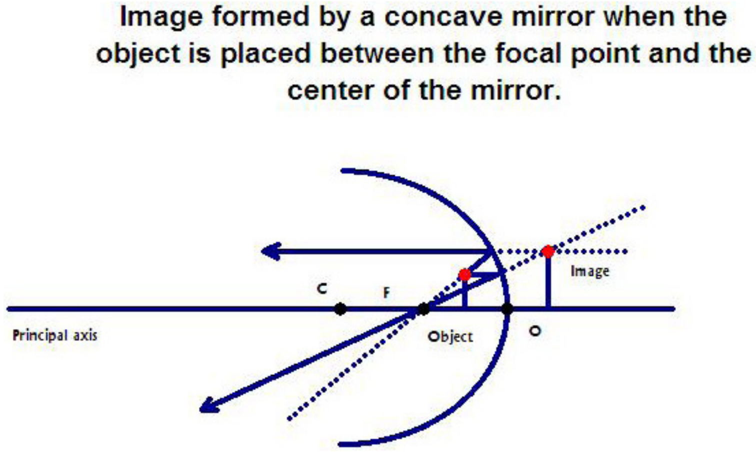

Figure 17.5

The image formed by a concave mirror when the object is placed between the focus and the center of the mirror has the following properties.

1. The image is virtual.
2. The image is erect.
3. The image is enlarged.
4. The image is located behind the mirror.

Convex Mirror

A *convex mirror* is a spherical mirror with the outside surface being the reflecting surface. The following diagram shows a convex mirror.

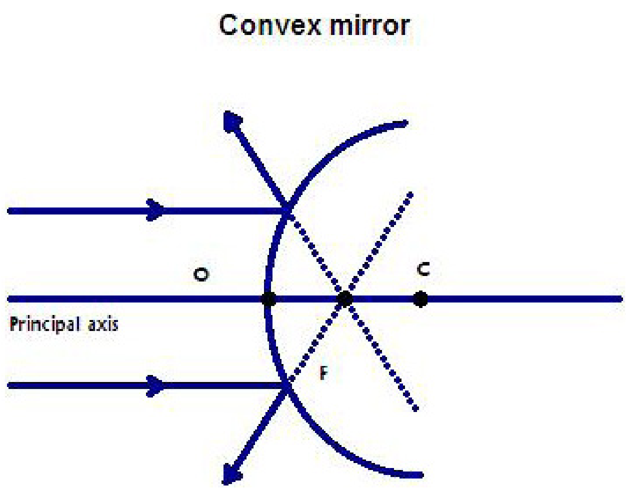

Figure 17.6

The center of the spherical surface is called the *center of curvature* (point C on the diagram) of the mirror. The mid-point of the mirror is called the *center of the mirror* (point O on the diagram). The line joining the center of curvature and the center of the mirror is called the *principal axis* of the mirror. The point from which light rays parallel to the principal axis seem to come from after reflection is called the *focus* (point F on the diagram) of the mirror. The focus of a convex mirror is virtual because actual light rays do not meet at the focus. The focus is located midway between the center of curvature and the center of the mirror. The distance between the focus and the center of the mirror is called the *focal length* (f) of the mirror.

There are *3* special light rays that can be used in constructing images formed by a convex mirror.

1. A light ray parallel to the principal axis seems to come from the focus after reflection.
2. A light ray directed towards the focus is reflected back parallel to the principal axis.
3. A light ray directed towards the center of curvature is reflected in its own path.

The following diagram shows the image construction of the image formed by a convex mirror.

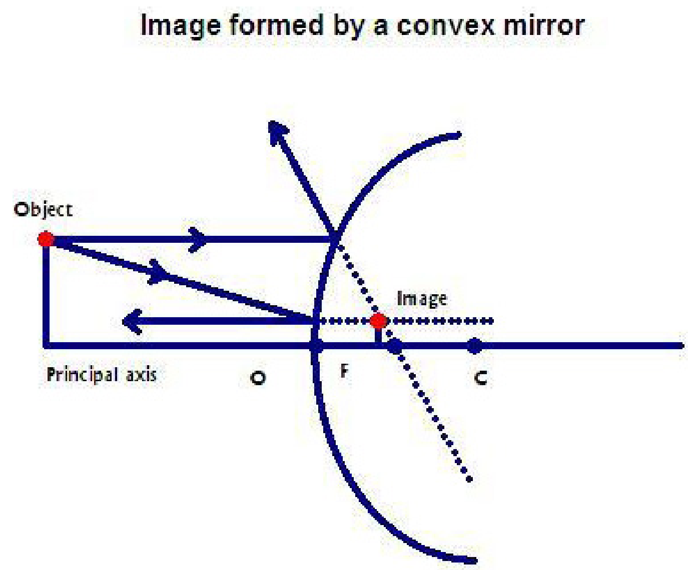

Figure 17.7

The image formed by a convex mirror has the following properties.

1. The image is virtual.
2. The image is erect.
3. The image is diminished.
4. The image is located behind the mirror.

The Mirror Equation

The mirror equation is an equation that relates the distance of the object from the center of the mirror, the distance of the image from the center of the mirror and the focal length (distance between focus and the center of the mirror). The distance between the object and the center of the mirror is called *object distance* (p). It is taken to be positive if the object is real and negative if the object is virtual. A virtual object is possible when more than one mirrors are involved. The distance between the image and the center of the mirror is called the *image distance* (q). The image distance is taken to be positive if the image is real and negative if the image is virtual. The focal length (f) is taken to be positive if the focus is real and negative if the focus is virtual. Thus, the focal length of a concave mirror is positive since its focus is real and that of a convex mirror is negative because its focus is virtual. The focal length of a mirror is half the radius of curvature: $|f| = R/2$ where R is the radius of curvature of the mirror. The following equation is the mirror equation.

$$1/f = 1/p + 1/q$$

The *magnification* (M) of a mirror is defined to be the ratio between the size of the image (h_i) and the size of the object (h_o). The size of the object (image) is taken to be positive if the object (image) is erect and negative if the object (image) is inverted.

$$M = h_i/h_o$$

It can also be shown that the magnification is equal to the negative of the ratio between image distance and object distance.

$$M = -q/p$$

Example: An object of height 0.02 m is placed 0.4 m in front of a concave mirror whose radius of curvature is 0.1 m.

a) Determine its focal length.

Solution: The focal length of a concave mirror is positive.

$R = 0.1$ m; $f = ?$

$|f| = R/2 = 0.1/2$ m $= 0.05$ m
$f = 0.05$ m

b) Calculate the distance of the image from the mirror.

Solution: $p = 0.4$ m; $q = ?$

$1/f = 1/p + 1/q$

$1/q = 1/f - 1/p = (1/0.05 - 1/0.4)$ 1/m $= 17.5$ 1/m

$q = 1/17.5$ m $= 0.06$ m

c) Is the image real or virtual?

Solution: The image is real because the image distance is positive.

d) Calculate the magnification.

Solution: $M = ?$

$M = -q/p = -0.06/0.4 = -0.15$

e) Calculate the size of the image.

Solution: $h_o = 0.02$ m; $h_i = ?$

$M = h_i/h_o$

$h_i = Mh_o = -0.15 * 0.02$ m $= 0.-003$ m

f) Is the image erect or inverted?

Solution: The image is inverted because h_i is negative.

Practice Quiz 17.1

Choose the best answer

1. Which of the following is a correct statement?
   A. The image formed by a flat mirror is real.
   B. The image formed by a flat mirror has the same size as the object.
   C. The image formed by a flat mirror is inverted in a direction parallel to the mirror.
   D. For an image formed by a flat mirror, the perpendicular distance between the image and mirror is less than the perpendicular distance between the object and the mirror.
   E. The image formed by a flat mirror is not inverted in a direction perpendicular to the mirror.

2. Which of the following is a correct statement?
   A. The focal point of a convex mirror is the point from which light rays parallel to the principal axis seem to come from after reflection.
   B. The center of curvature of a concave mirror is located midway between the focal point and the center of the mirror.
   C. The focal point of a concave mirror is the point from which light rays parallel to the principal axis seem to come from after reflection.
   D. The focal point of a concave mirror is virtual.
   E. The focal point of a convex mirror is real.

3. Which of the following is a correct statement about a concave mirror?
   A. All of the other choices are not correct.
   B. A light ray parallel to the principal axis is reflected back through the center of curvature.
   C. A light ray through the center of curvature of the mirror returns in its own path after reflection.
   D. A light ray through the focal point returns in its own path.
   E. A light ray directed to the center of the mirror returns in its own path.

4. Which of the following is true about a convex mirror.
   A. A light ray directed the center of the mirror is reflected in its own path.
   B. The focal point of a convex mirror is real.
   C. A light ray parallel to the principal axis seems to come from the center of curvature after reflection.
   D. A light ray directed towards the center of curvature is reflected in its own path.
   E. A light ray directed towards the focal point is reflected in its own path.

5. When an object is placed between the center of curvature and the focal point of a concave mirror
   A. the image is enlarged.
   B. the image is erect.
   C. the image is virtual
   D. All of the other choices are not correct.
   E. the image is formed behind the mirror.

6. When an object is placed between a concave mirror and its focal point,
   A. none of the other choices are correct.
   B. the image is real.
   C. the image is formed in front of the mirror.
   D. the image is inverted.
   E. the image is enlarged.

7. When an object is placed beyond the center of curvature (at a distance greater than the radius) of a concave mirror,
   A. none of the other choices are correct.
   B. the image is formed between the mirror and the focal point.
   C. the image is real.
   D. the image is enlarged
   E. the image is erect.

8. For an object placed in front of a convex mirror,
   A. the image is formed in front of the mirror.
   B. the image may be erect or diminished.
   C. the image may be real or virtual
   D. the image is always real
   E. the image is always diminished.

9. An object is placed *0.2* m in front of a concave mirror whose radius of curvature is *0.06* m. Calculate the image distance.
   A. *3.176e-2* m
   B. *3.529e-2* m
   C. *4.941e-2* m
   D. *4.588e-2* m
   E. *2.824e-2* m

10. An object is placed *0.08* m in front of a convex mirror whose radius of curvature is *0.08* m. Calculate the image distance.
    A. *-2.133e-2* m
    B. *-3.733e-2* m
    C. *-3.467e-2* m
    D. *-2.667e-2* m
    E. *-2.4e-2* m

11. An object of height *0.025* m is placed *0.12* m in front of a concave mirror whose radius of curvature is *0.12* m. Calculate the height of the image.
    A. *-1.75e-2* m
    B. *-2.5e-2* m
    C. *-2e-2* m
    D. *-3.5e-2* m
    E. *-2.75e-2* m

12. An object of height *0.04* m is placed *0.16* m in front of a convex mirror whose radius of curvature is *0.06* m. Calculate the height of the image.
    A. *0.632e-2* m
    B. *0.379e-2* m
    C. *0.442e-2* m
    D. *0.505e-2* m
    E. *0.695e-2* m

Lenses

A *lens* is a piece of glass with spherical surfaces. There are two types of lenses. They are convex (converging) lens and concave (diverging) lens.

Convex Lens

The following diagram shows a convex lens.

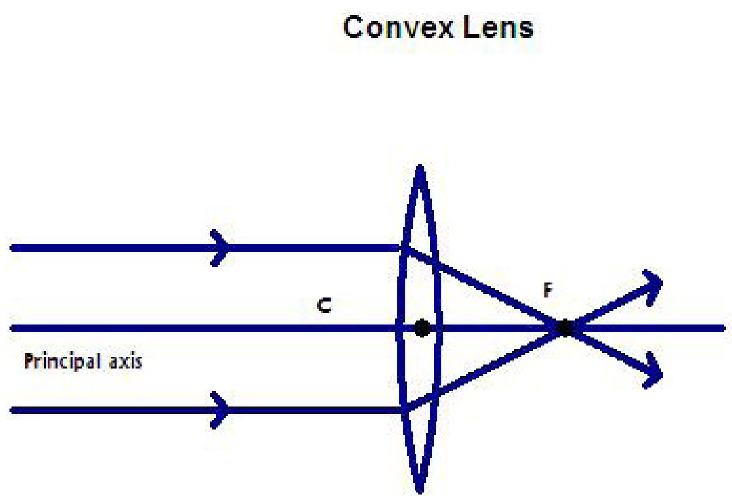

Figure 17.8

The mid-point of the lens (point C in the diagram) is called the *center of the lens*. The line joining the centers of curvature of both surfaces and the center of the lens is called the *principal axis* of the lens. The point at which light rays parallel to the principal axis converge after refraction is called the *focus* (point F on the diagram) of the lens. The focus of a convex lens is real because actual light rays meet at the point. The distance between the focus and the center of the lens is called the *focal length* of the lens.

There are three special light rays that can be used to construct images formed by a convex lens.

1. A light ray parallel to the principal axis passes through the focal point after refraction.
2. A light ray through the focal point is refracted parallel to the principal axis.
3. A light ray through the center of the lens passes undeflected.

The following diagram shows the image construction of the image formed by a convex lens when the object is placed beyond twice the focal length.

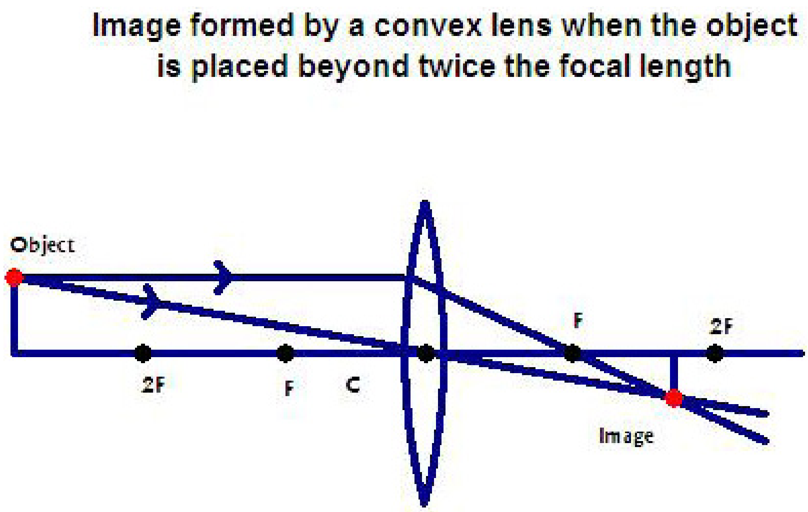

Figure 17.9

The image formed by a convex lens when the object is placed beyond twice the focal length has the following properties.

1. The image is real.
2. The image is inverted.
3. The image is diminished.
4. The image is located at a distance greater than the focal length but smaller than twice the focal length on the other side of the lens.

The following diagram shows the image construction of the image formed by a convex lens when the object is located at a distance greater than the focal length but less than twice the focal length.

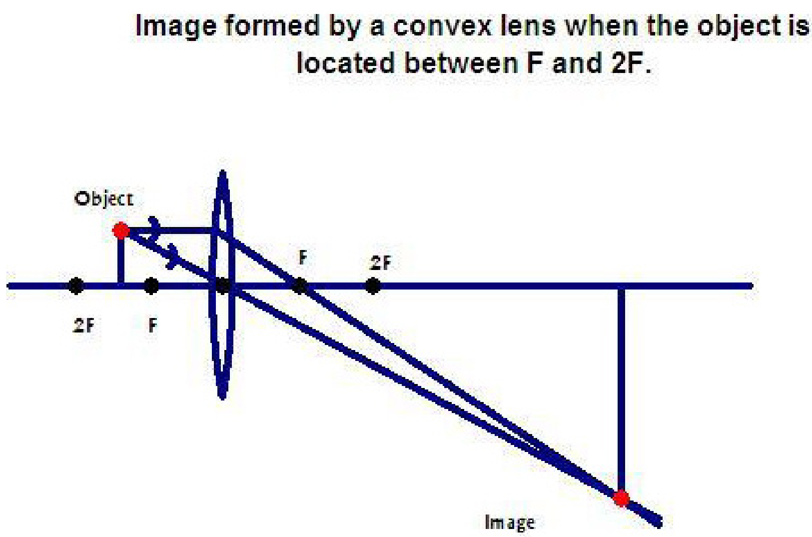

Figure 17.10

The image formed by a convex lens when the object is placed at a distance greater than the focal length but less than twice the focal length has the following properties.

1. The image is real.
2. The image is inverted.
3. The image is enlarged.
4. The image is located beyond twice the focal length on the other side of the lens.

The following diagram shows the image construction of the image formed by a convex lens when the object is placed between the focus and the center of the lens.

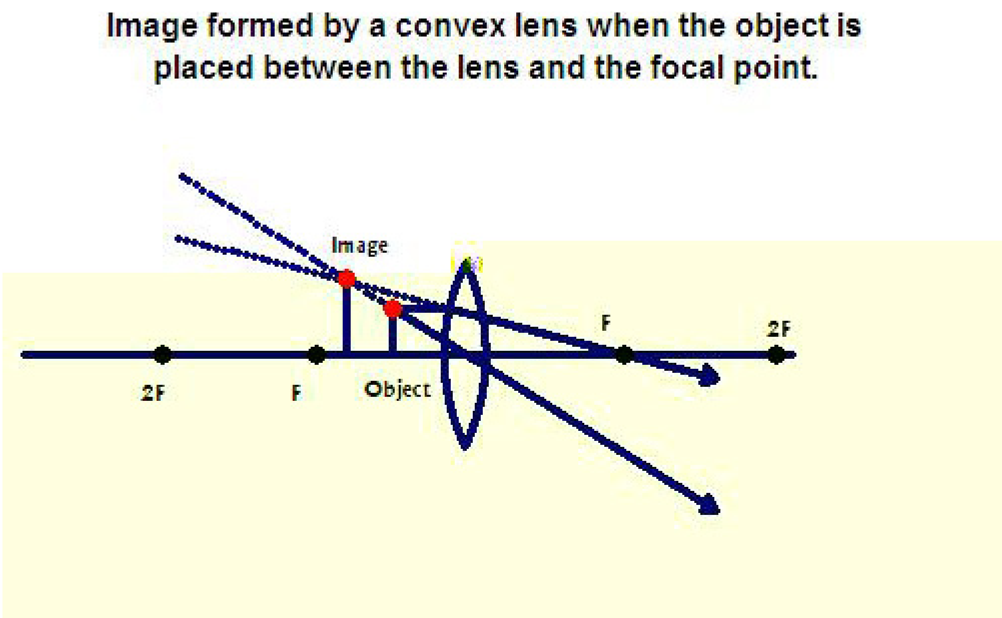

Figure 17.11

The image formed by a convex lens when the object is located between the focus and the center of the lens has the following properties.

1. The image is virtual.
2. The image is erect.
3. The image is enlarged.
4. The image is formed on the same side as the object.

Concave Lens

The following diagram shows a concave lens.

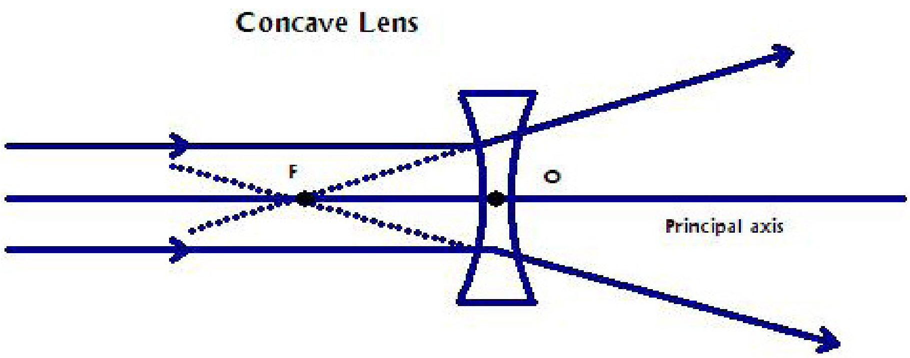

Figure 17.12

The mid-point of the lens (point O in the diagram) is called the *center of the lens*. The line joining the centers of curvature of the surfaces of the lens and the center of the lens is called the *principal axis* of the lens. The point from which light rays parallel to the principal axis seem to come from after refraction is called the *focus* of the lens. The focus of a concave lens is virtual because actual light rays do not meet at the focus. The distance between the focus and the center of the lens is called the *focal length* of the lens.

There are three special light rays used to construct images formed by a concave lens.

1. A light ray parallel to the principal axis seems to come from the focal point after refraction.
2. A light ray directed towards the focal point is refracted parallel to the principal axis.
3. A light ray directed towards the center of the lens passes undeflected.

The following diagram shows the image construction of the image formed by a concave lens.

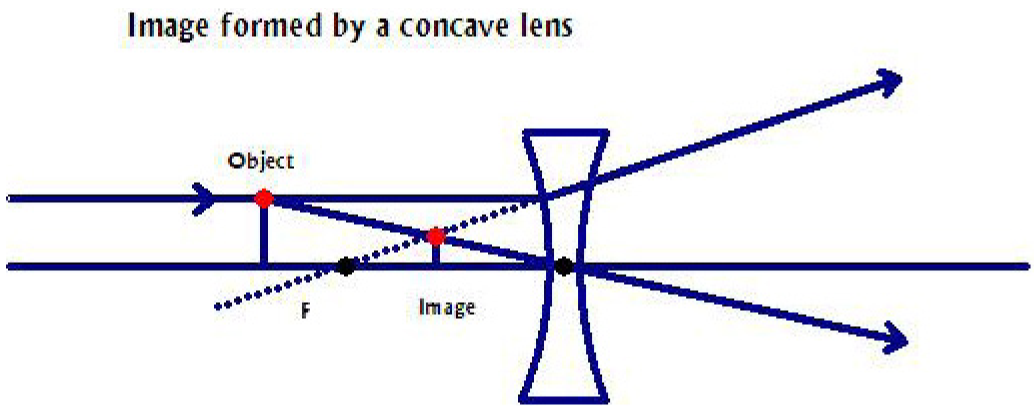

Figure 17.13

The image formed by a concave lens has the following properties.

1. The image is virtual.
2. The image is erect.
3. The image is diminished.
4. The image is located on the same side as the object.

The Lens Equation

The lens equation is an equation that relates the object distance, image distance and the focal length. The *object distance* (p) is the distance between the object and the center of the lens. It is taken to be positive if the object is real and negative if the object is virtual. The *image distance* (q) is the distance between the image and the center of the lens. It is taken to be positive if the image is real and negative if the image is virtual. The focal length (f) is the distance between the focal point and the center of the lens. The focal length is taken to be positive if the focus is real and negative if the focus is virtual. This means the focal length of a convex lens is positive (because its focus is real) and that of a concave lens is negative (because its focus is virtual). The following equation is the so called lens equation.

$$1/f = 1/p + 1/q$$

The magnification of a lens is defined to be the ratio between the size of the image (h_i) and the size of the object (h_o). The size of the object (image) is taken to be positive if the object (image) is erect and negative if the object (image) is inverted.

$$M = h_i/h_o$$

The magnification is also equal to the negative of the ratio between the image distance and object distance.

$$M = -q/p$$

Example: An object of of height *0.03* m is placed *0.3* m in front of a concave lens whose focal length is *0.06* m.

a) Calculate the distance of the image from the center of the lens.

Solution: The focal length is negative because the lens is concave.

$f = -0.06$ m; $p = 0.3$ m; $q = ?$

$$1/q = 1/f - 1/p = (1/(-0.06) - 1/0.3)\ \text{m}^{-1} = -20\ \text{m}^{-1}$$

$$q = 1/(-20)\ \text{m} = -0.05\ \text{m}$$

b) Is the image real or virtual?

Solution: It is virtual because the image distance is negative.

c) Calculate its magnification.

Solution: $M = ?$

$$M = -q/p = - -0.05/0.3 = 0.17$$

d) Calculate the height of the image and determine if the image is erect or inverted.

Solution: $h_o = 0.03$ m; $h_i = ?$

$$M = h_i/h_o$$

$$h_i = Mh_o = 0.17 * 0.03\ \text{m} = 0.0051\ \text{m}$$

The image is erect because h_i is positive.

Lens Makers Equation

If the radius of curvature of the surface of a lens upon which the light rays are incident is R_1 and the radius of curvature of the other surface is R_2, then the focal length of the lens is given by

$$1/f = (n - 1)\,(1/R_1 - 1/R_2)$$

Where n is the refractive index of the lens. A radius of curvature of the surface of a lens is taken to be positive if the direction from the surface towards the center of curvature of the surface is the same as the direction of the incident light rays and negative if opposite to the direction of the incident light rays.

Example: Both surfaces of a convex lens have a radius of curvature of 0.05 m. The refractive index of the glass is 1.5. Calculate the focal length of the lens.

Solution: The radius of curvature of the surface upon which the light rays are incident (R_1) is positive because the direction from the surface towards its center of curvature is the same as the direction of the incident light rays. The radius of curvature of the other surface is negative because the direction from the surface to its center of curvature is opposite to the direction of the incident light rays.

$n = 1.5$; $R_1 = 0.05$ m; $R_2 = -0.05$ m; $f = ?$

$$1/f = (n - 1)\,(1/R_1 - 1/R_2) = (1.5 - 1) * (1/0.05 - 1/{-0.05})\ \mathrm{m}^{-1} = 1/0.05\ \mathrm{m}^{-1}$$

$$f = 0.05\ \mathrm{m}$$

Example: Both surfaces of a concave lens have a radius of curvature of 0.04 m. The refractive index of the glass is 1.5. Calculate the focal length of the lens.

Solution: The radius of curvature of the surface upon which the light rays are incident (R_1) is negative because the direction from the surface towards its center of curvature is opposite to the direction of the incident light rays. The radius of curvature of the other surface is positive because the direction from the surface to its center of curvature is the same as the direction of the incident light rays.

$n = 1.5$; $R_1 = -0.04$ m; $R_2 = 0.05$ m; $f = ?$

$$1/f = (n - 1)\,(1/R_1 - 1/R_2) = (1.5 - 1) * (1/{-0.04} - 1/0.04)\ \mathrm{m}^{-1} = -1/0.04\ \mathrm{m}^{-1}$$

$$f = -0.04\ \mathrm{m}$$

Practice Quiz 17.2

Choose the best answer

1. Which of the following is a correct statement?
   A. None of the other choices are correct.
   B. The focal point of a convex lens is virtual.
   C. The focal point of a concave lens is real.
   D. The focal point of a convex lens is the point from which light rays parallel to the principal axis seem to come from after refraction.
   E. The focal point of a concave lens is the point from which light rays parallel to the principal axis seem to come from after refraction.

2. Which of the following is a correct statement?
   A. A light ray parallel to the principal axis of a concave lens is refracted through the focal point of the lens.
   B. A light ray through the focal point of a concave lens is refracted parallel to the principal axis
   C. All of the other choices are not correct.
   D. A light ray through the focal point of a convex lens passes undeflected.
   E. A light ray parallel to the principal axis of convex lens is refracted through the focal point.

3. When an object is placed at a distance greater than the focal length but less than twice the focal length from a convex lens
   A. the image is virtual.
   B. All of the other choices are not correct.
   C. the image is formed on the same side as the object.
   D. the image is inverted.
   E. the image is diminished.

4. When an object is placed between a convex lens and its focal point,
   A. the image is formed on the side of the lens as the object.
   B. the image is inverted.
   C. the image is diminished
   D. the image is real.
   E. None of the other choices are correct.

5. When an object is placed at a distance greater than twice the focal length from a convex lens,
   A. the image is erect.
   B. the image is real.
   C. the image is enlarged
   D. None of the other choices are correct.
   E. the image is formed between the lens and the focal point of the lens.

6. For an object placed in front of a concave lens,
   A. the image is formed on the same side as the object.
   B. the image is always enlarged.
   C. the image is always inverted.
   D. the image may be real or virtual
   E. the image is always real

7. An object of height *0.02* m is placed *0.18* m in front of a convex lens whose focal length is *0.03* m. Calculate the height of the image.
   A. *-0.52e-2* m
   B. *-0.44e-2* m
   C. *-0.32e-2* m
   D. *-0.4e-2* m
   E. *-0.24e-2* m

8. An object of height *0.03* m is placed *0.2* m in front of a concave lens whose focal length is *0.04* m. Calculate the magnification.
   A. *0.133*
   B. *0.183 0.233*
   C. *0.217*
   D. *0.167*

9. The two radii of curvatures of a convex lens, made of glass, are *0.05* m and *0.02* m. Calculate the focal length of the lens. (Refractive index of glass is 1.5).
   A. *-7.333e-2* m
   B. *-2.857e-2* m
   C. *2.857e-2* m
   D. *-6.667e-2* m
   E. *6.667e-2* m

10. The two radii of curvatures of a concave lens, made of glass, are *0.07* m and *0.1* m. Calculate the focal length of the lens. (Refractive index of glass is 1.5).
    A. *-8.235e-2* m
    B. *42e-2* m
    C. *8.235e-2* m
    D. *-42e-2* m
    E. *46.667e-2* m

18 WAVE PROPERTIES OF LIGHT

Your goals for this chapter are to learn about interference of light, diffraction of light, and polarization of light.

Interference of Light

Interference of Light is the meeting of two or more light waves at the same point at the same time. The net instantaneous effect of the interfering waves is obtained by adding the instantaneous values of the waves algebraically. The net effect of the interfering waves $y_1 = A_1 \cos(\omega t - kx_1)$ and $y_2 = A_2 \cos(\omega t - kx_2)$ is given as $y_{net} = y_1 + y_2 = A_1 \cos(\omega t - kx_1) + A_2 \cos(\omega t - kx_2)$.

Constructive interference is interference with the maximum possible effect. For light waves, constructive interference results in a bright spot. The amplitude of the net wave of two interfering waves is equal to the sum of the amplitudes of the interfering waves. It occurs when the phase shift (δ) between the interfering waves is an integral multiple of 2π. The following equation is the condition for constructive interference.

$$\delta_n = 2n\pi$$

Where n is an integer; that is, n is a member of the set $\{ \ldots -2, -1, 0, 1, 2, \ldots \}$ and δ_n is a member of the set $\{ \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots \}$.

Destructive interference is interference with the minimum possible effect. For light waves, destructive interference results in a dark spot. The amplitude of the net wave of two interfering waves is equal to the difference between the amplitudes of the interfering waves. It occurs when the phase shift between the interfering waves is an odd-integral multiple of π. The following equation is the condition for destructive interference.

$$\delta_n = (2n + 1)\pi$$

Where n is integer; that is n is a member of the set $\{ \ldots -2, -1, 0, 1, 2, 3, \ldots \}$ and δ_n is a member of the set $\{ \ldots -3\pi, -\pi, \pi, 3\pi, \ldots \}$.

Example: Determine if the following waves will interfere constructively, destructively, or neither constructively nor destructively.

a) $y_1 = 5 \cos (20t + \pi)$ and $y_2 = 7 \cos (20t + 4\pi)$.

Solution: $\beta_1 = \pi$; $\beta_2 = 4\pi$; $\chi = ?$

$$\delta = \beta_2 - \beta_1 = 4\pi - \pi = 3\pi$$

The two waves will interfere destructively because $\delta = 3\pi$ is a member of the set $\{ \ldots -3\pi, -\pi, \pi, 3\pi, \ldots \}$

b) $y_1 = 30 \cos (40t + \pi/2)$ and $y_2 = 80 \cos (40t + 7\pi)$.

Solution: $\beta_1 = \pi/2$; $\beta_2 = 7\pi$; $\delta = ?$

$$\delta = \beta_2 - \beta_1 = 7\pi - \pi/2 = 13\pi/2$$

The two waves will interfere neither constructively nor destructively because $\delta = 13\pi/2$ is not a member of the set $\{ \ldots -3\pi, -\pi, \pi, 3\pi, \ldots \}$ or the set $\{ \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots \}$

c) $y_1 = 2\cos(50t - 5\pi/2)$ and $y_2 = 80\cos(50t - \pi/2)$.

Solution: $\beta_1 = -5\pi/2$; $\beta_2 = -\pi/2$; $\delta = ?$

$$\delta = \beta_2 - \beta_1 = -\pi/2 - -5\pi/2 = 2\pi$$

The two waves will interfere constructively because $\delta = 2\pi$ is a member of the set $\{\ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots\}$

The conditions of constructive and destructive interference can also be expressed in terms of the path difference (difference between the distances travelled by the two waves) between the two waves. If the two interfering waves are given as $y_1 = A_1 \cos(\omega t - kx_1)$ and $y = A_2 \cos(\omega t - kx_2)$ (where $k = 2\pi/\lambda$), then the phase shift between the two waves is $\delta = 2\pi x_2/\lambda - 2\pi x_1/\lambda = (2\pi/\lambda)(x_2 - x_1) = (2\pi/\lambda)\Delta$ where $\Delta = x_2 - x_1$ is the path difference between the two waves. The condition of constructive interference may be written in terms of path difference as $\delta_n = 2n\pi = (2\pi/\lambda)\Delta_n$. This implies that the path difference between two waves has to satisfy the following condition for constructive interference.

$$\Delta_n = n\lambda$$

Where n is an integer; that is n is a member of the set { … -2, -1, 0, 1, 2, … } and Δ_n is a member of the set $\{\ldots -2\lambda, -\lambda, 0, \lambda, 2\lambda, \ldots\}$. Two waves will interfere constructively if their path difference is an integral multiple of the wavelength of the waves.

The condition of destructive interference can be written in terms of path difference as $\delta_n = (2n + 1)\pi = (2\pi/\lambda)\Delta_n$. This implies that the path difference between two waves has to satisfy the following condition if the waves are to interfere destructively.

$$\Delta_n = (n + 1/2)\lambda$$

where n is integer; that is n is a member of the set $\{\ldots -2, -1, 0, 1, 2, \ldots\}$ and Δ_n is a member of the set $\{\ldots -3\lambda/2, -\lambda/2, \lambda/2, 3\lambda/2, \ldots\}$. Two waves will interfere destructively if their path difference is half-odd-integral multiple of the wavelength.

Diffraction of Light

Diffraction of light is the bending of light as light encounters an obstacle. Light travels in a straight line. But when light encounters an obstacle it scatters in all directions. When light is blocked by an opaque object, it can be still seen behind the opaque object because of diffraction of light at the edges of the object. A large part of a room can be seen through a key hole even though light travels in straight lines. This is because of diffraction of light at the key hole which bends the light. This property of light enables one to use a narrow slit as a source of light because as light crosses the slit it is scattered in all directions.

Young's Double Slit Experiment

The following diagram shows the setup of Young's double slit experiment.

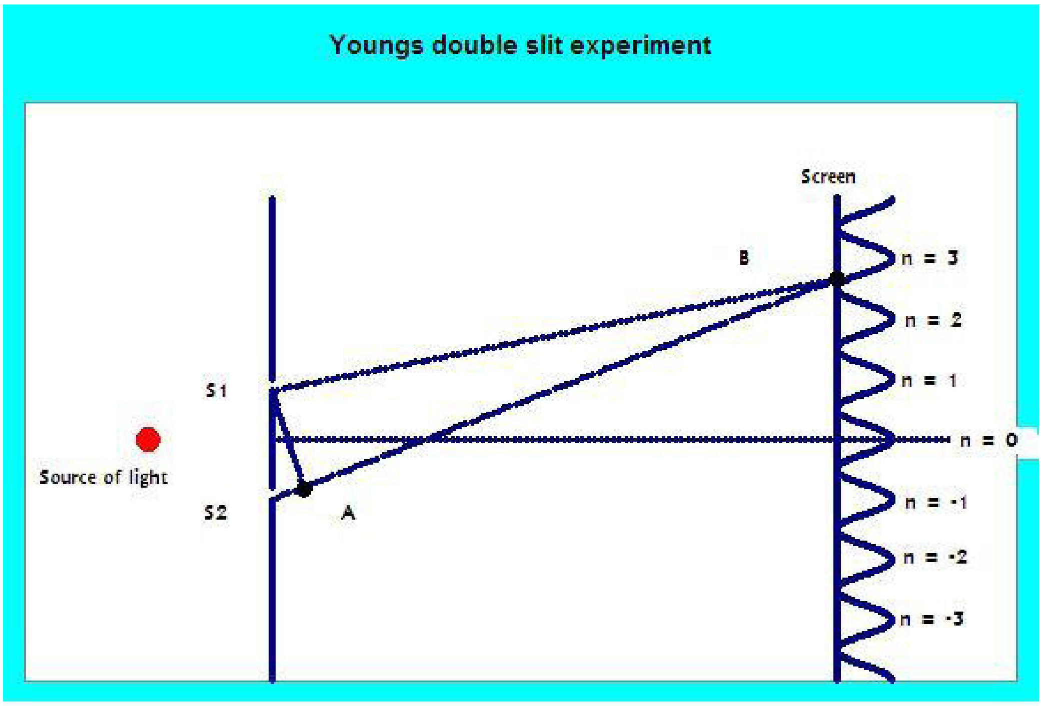

Figure 10.1

Young's double slit experiment consists of an opaque material with two very narrow slits (s1 and s2 in the diagram) and a screen at some distance from this material. When the opaque material is exposed to a source of light, the two slits serve as two sources of light because light is diffracted in all directions as it passes through the slits. The light waves from the two slits interfere on the screen. The diagram shows light waves from the two slits interfering at point B. The experiment shows that bright (constructive interference) and dark (destructive interference) spots appear alternatively on the screen. The graph on the screen is a representation of the intensity of light observed on the screen. The interference pattern observed on the screen depends on the path difference between the two waves. If the path difference is an integral multiple of the wavelength of the light, the two waves interfere constructively and a bright spot is observed. If the path difference is half-odd-integral multiple of the wavelength of the light, destructive interference takes place and a dark spot is observed. At the center of the screen the two waves travell the same distance and the path difference is zero which implies constructive interference and a bright spot is observed. This corresponds to $n = 0$ and is called the zeroth order. As one goes further from the center, the path difference between the waves increases. At a point where the path difference is half of the wavelength, destructive interference takes place and a dark spot is observed. This way, as the path difference alternates between integral multiples of the wavelength and half-odd-integral multiples of the wavelength, the interference pattern alternates between bright spots and dark spots. The n^{th} bright spot is called the n^{th} order bright spot.

The path difference between the waves from slit s1 and slit s2 can be obtained by dropping the perpendicular from slit s1 to the light wave from slit s2 (the line joining slit s1 and point A in the diagram). Then the path difference (δ) is the distance between slit s2 and point A. If the distance between the slits is d and the angle formed between the line joining the slits and the line joining slit s1 and point A is θ (This angle is also equal to the angle formed by the line joining the midpoint of the slits to point B with the horizontal), then the path difference is given as $\delta = d \sin(\theta)$. Therefore the condition for constructive interference (bright spot) for Young's double slit experiment is

$$d \sin(\theta) = n\lambda$$

Where n is an integer and λ is the wavelength of the light. Similarly, the condition for destructive interference (dark spot) is

$$d \sin(\theta) = (n + 1/2)\lambda$$

If the perpendicular distance between the source and the screen is D, then the vertical distance (y) between the the center of the screen (zeroth order bright spot) and an order corresponding to an angle θ is given by

$$y = D \tan(\theta)$$

Example: In Young's double slit experiment, the slits are separated by a distance of $2e\text{-}6$ m. The second order bright spot is observed at an angle of $26°$.

a) Calculate the wavelength of the light.

Solution: $d = 2e\text{-}9$ m; $\theta = 26°$; $n = 2$; $\lambda = ?$

$$d \sin(\theta) = n\lambda$$

$$\lambda = d \sin(\theta)/2 = 2e\text{-}6 * \sin(26°)/2 \text{ m} = 4.4e\text{-}9\ m$$

b) If the perpendicular distance between the source and the screen is 0.05 m, calculate the distance between the zeroth order bright spot and the second order bright spot on the screen.

Solution: $D = 0.05$ m; $\theta = 26°$; $y = ?$

$$y = D \tan(\theta)$$

$$= 0.05 * \tan(26°) = 0.024 \text{ m}$$

Thin Film Interference

When light of wavelength λ is incident on a thin film of refractive index n, some of the light rays will be reflected from the upper surface, and some of the light rays will be refracted to the lower surface and reflected from the lower surface and then refracted back to air. These two light rays will interfere and create interference pattern. There are two factors that contribute to the phase difference between the two waves:

a) the path difference between the two waves. If the thickness of the film is t and the incident light rays are approximately perpendicular to the surface, the path difference is $2t$. Since the wavelength of the light in the film is λ/n, this corresponds to a phase difference of $2\pi(2t/(\lambda/n))$ radians = $2\pi(2tn/\lambda)$ radians.
b) the phase difference due to phase shift on reflection from an optically denser medium. The light rays reflected from the upper surface will have a phase shift of π radians because they are being reflected from an optically denser medium (film) while the light rays reflected from the lower surface will not have a phase shift, because they are being reflected from an optically less dense medium (air). Thus, the two waves will have a phase shift of π radians due to reflection.

Therefore the net phase difference between the two waves is $2\pi(2nt/\lambda) + \pi$. This implies that the condition for constructive interference is $2\pi(2nt/\lambda) + \pi = 2m\pi$ where m is a natural number, or

$$2nt = (m - 1/2)\,\lambda$$

Similarly, the condition for destructive interference is $2\pi(2nt/\lambda) + \pi = (2m + 1)\,\pi$ where m is a natural number, or

$$2nt = m\,\lambda$$

Example: Calculate the thickness of a thin film of refractive index 1.3 that results in a second order bright spot, when light rays of wavelength $7e\text{-}7$ m are incident on the film approximately perpendicularly.

Solution: $n = 1.3$; $\lambda = 7e\text{-}7$ m; $m = 2$; $t = ?$

$$2nt = (m - 1/2)\,\lambda$$

$$t = (m - 1/2)\,\lambda/(2n) = (2 - 1/2) * 7e\text{-}9/(2 * 1.3) \text{ m} = 4.038e\text{-}7 \text{ m}.$$

Single Slit Diffraction

When light enters a slit whose width is of the same order as the wavelength of the light, it will be diffracted and each point of the slit can be considered as a source of light. The light waves from each point of the slit meet on a screen and form interference pattern. The interference of all of the waves can be regrouped into pairs of waves and then added. Imagine the slit being divided into half. If the width of the slit is a, for every wave in the lower half, there is a wave in the upper half at a distance of $a/2$. If the waves are grouped into pairs of waves with the distance between their sources being $a/2$, then the condition for destructive (constructive) interference becomes the same for all of the pairs, and thus the condition of interference can be applied to one of the pairs only. As discussed earlier, if the distance between the sources is $a/2$, then the path difference between the waves is $a \sin(\theta)/2$ where θ is defined in the same way as above. Therefore there will be destructive interference, if the path difference is equal to half of the wavelength or if $a \sin(\theta) = \lambda$. The same argument can be repeated by imagining the slit to be divided into $2, 4, .. N$ parts and regrouping the waves into pairs of waves whose sources are separated by a distance of a/N and the following general formula for destructive interference can be obtained.

$$a \sin(\theta) = n\lambda$$

Where n is an integer.

Diffraction Grating

A *diffraction grating* is a piece of glass with a lot of slits spaced uniformly. Again we can imagine the waves being grouped into pairs before being added. If each slit is grouped with the slit next to it, then the distance between the slits for each pair of waves will be the same (which is the distance between two neighboring slits). This means the condition of interference is the same for all of the pairs of waves. As a result the condition of interference can be applied to one of the pairs only. If the slits are separated by a distance d, then the path difference between the waves of a pair is $d \sin(\theta)$ (with θ as defined above). Therefore, the conditions of constructive and destructive interference respectively are

$$d \sin(\theta) = n\lambda$$

$$d \sin(\theta) = (n + 1/2)\lambda$$

Where n is an integer.

Example: A diffraction grating has *10000* slits. Its width is *0.02* m. When a certain light wave is diffracted through it, the first bright spot was observed at an angle of *16*°. Calculate, the wavelength of the light.

Solution: The separation between the slits (d) may be obtained by dividing the width of the diffraction grating by the number of slits.

Width = *0.02* m, number of slits = *1000*; $d = 0.02/10000$ m = *2e-6* m; $\theta = 16°$; $n = 1$; $\lambda = ?$

$$d \sin(\theta) = n\lambda$$

$$\lambda = d \sin(\theta)/n = 2e\text{-}6 * \sin(16°)/n = 5.5e\text{-}7 \text{ m}$$

Polarization

Electromagnetic waves (light is an electromagnetic wave) are transverse waves which means the electric and magnetic fields are perpendicular to the direction of propagation of energy. This limits the direction of the electric field to the plane perpendicular to the direction of propagation of energy. But it can have any direction on that plane. Normally light will include electric fields with all of the possible directions (because light is produced with charges accelerating in no preferred direction). Such kind of light is called an unpolarized light. But when light passes through some materials (devices) called polarizers, there will be a preferred direction and the electric field will vibrate in a certain fixed direction. Such light where the electric field fibrates only in a certain fixed direction is called *linearly polarized light*.

Polarized light may be created by selective absorption. There are some materials whose molecules vibrate only in a certain direction. When light passes through such kind of materials, the component of the electric field in the direction of vibration of the molecules will be absorbed because it will be used to accelerate the charges. Only the perpendicular component (which has a unique direction because to start with the directions were limited to a plane) will pass through unaffected. As a result the outcome is light where the electric field vibrates in a fixed direction which is linearly polarized light.

Polarized light also may be created by reflection. When light is incident on a boundary between two mediums some of the light will be refracted and some of the light will be reflected. The component of the electric field parallel to the surface and the component perpendicular to the surface reflect differently. The component parallel to the surface reflects more strongly. In fact, at a certain angle of incidence θ_p, where the reflected ray and the refracted ray are perpendicular to each other, the refracted ray will contain only the component parallel to the surface resulting in a polarized light. For this special incident rays, the sum of the angle of incidence (θ_p) and angle of refraction (θ_r) is $90°$. Or, $\theta_r = 90° - \theta_p$. Assuming the light ray is entering a medium of refractive index n_2 from a medium of refractive index n_1, $n_2 / n_1 = \sin(\theta_p) / \sin(90° - \theta_p) = \sin(\theta_p) / \cos(\theta_p) = \tan(\theta_p)$. This special angle of incidence that results in a purely polarized light is called *polarizing angle* and is given by

$$\theta_p = \arctan(n_2 / n_1)$$

This relationship is called *Brewster's law*.

Practice Quiz 18

Choose the best answer

1. Which of the following is a correct statement?
   A. When two waves interfere, the amplitude of the net wave is always equal to the sum of the amplitudes of the interfering waves.
   B. When two waves interfere, the amplitude of the net wave is always equal to the difference between the amplitudes of the interfering waves.
   C. Interference of two waves is the meeting of two waves at the same point at the same time.
   D. Diffraction is the bending of light as light crosses the boundary between two optical mediums.
   E. All of the other choices are correct statements.

2. Which of the following is a correct statement?
   A. Two waves of the same frequency interfere destructively only if their phase difference is an integral multiple of $180°$.
   B. Two waves interfere destructively only if their path difference is half of odd integral multiple of their wavelength.
   C. None of the other choices are correct.
   D. Two waves of the same frequency interfere constructively only if their path difference is an even integral multiple of their wavelength.
   E. Two waves of the same frequency interfere constructively, only if their phase difference is $0°$.

3. Which of the following pair of waves interfere constructively?
   A. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t + 4 * \pi)$
   B. $x = 10 \sin(30t + \pi/2)$ and $y = 20 \sin(30t + 7 * \pi/2)$
   C. $x = 10 \sin(30t + \pi/2)$ and $y = 20 \sin(30t + \pi)$
   D. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t + 5 * \pi)$
   E. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t)$

4. Which of the following pair of waves interfere destructively?
   A. $x = 10 \sin(30t + \pi/2)$ and $y = 20 \sin(30t + 5 * \pi/2)$
   B. $x = 10 \sin(30t + 2 * \pi)$ and $y = 20 \sin(30t)$
   C. $x = 10 \sin(30t + \pi/2)$ and $y = 20 \sin(30t + \pi)$
   D. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t + 4 * \pi)$
   E. $x = 10 \sin(30t + \pi)$ and $y = 20 \sin(30t + 5 * \pi)$

5. In Young's double slit experiment, the slits are separated by a distance of *3.28e-6* m. If the third order bright spot is formed at an angle of *38°*, calculate the wave length of the light.
   A. *942.372e-9* m
   B. *605.811e-9* m
   C. *471.186e-9* m
   D. *673.123e-9* m
   E. *538.499e-9* m

6. In Young's double slit experiment, the slits are separated by a distance of *3.26e-6* m. If the fourth order dark spot is formed at an angle of *32°*, calculate the wave length of the light.
   A. *537.456e-9* m
   B. *268.728e-9* m
   C. *383.897e-9* m
   D. *345.507e-9* m
   E. *307.118e-9* m

ANSWERS TO PRACTICE QUIZZES

Practice Quiz 12.1

1. D 2. E 3. B 4. B 5. C 6. B 7. A 8. C

Practice Quiz 12.2

1. B 2. D 3. D 4. D 5. E 6. E 7. E 8. D

Practice Quiz 13.1

1. C 2. C 3. C 4. C 5. D 6. B 7. D 8. E

Practice Quiz 13.2

1. E 2. B 3. E 4. E 5. B 6. A 7. B 8. A 9. C

Practice Quiz 14.1

1. D 2. D 3. A 4. B 5. C 6. A 7. C 8. A 9. B 10. E 11. A

Practice Quiz 14.2

1. E 2. A 3. A 4. B 5. D 6. C 7. D 8. B 9. A 10. A 11. B 11. C

Practice Quiz 15

1. D 2. C 3. B 4. C 5. E 6. C 7. C 8. C 9. A 10. E

Practice Quiz 16.1

1. C 2. B 3. D 4. A 5. D 6. B 7. A 8. B 9. D 10. D 11. A 12. B 13. A 14. B

Practice Quiz 16.2

1. A 2. B 3. C 4. B 5. C 6. C 7. B 8. B

Practice Quiz 17.1

1. B 2. A 3. C 4. D 5. A 6. E 7. C 8. E 9. B 10. D 11. B 12. A

Practice Quiz 17.2

1. E 2. E 3. D 4. A 5. B 6. A 7. D 8. E 9. C 10. A

Practice Quiz 18

1. C 2. B 3. D 4. D 5. D 6. C